# Spanning trees and logarithmic least squares optimality for complete and incomplete pairwise comparison matrices

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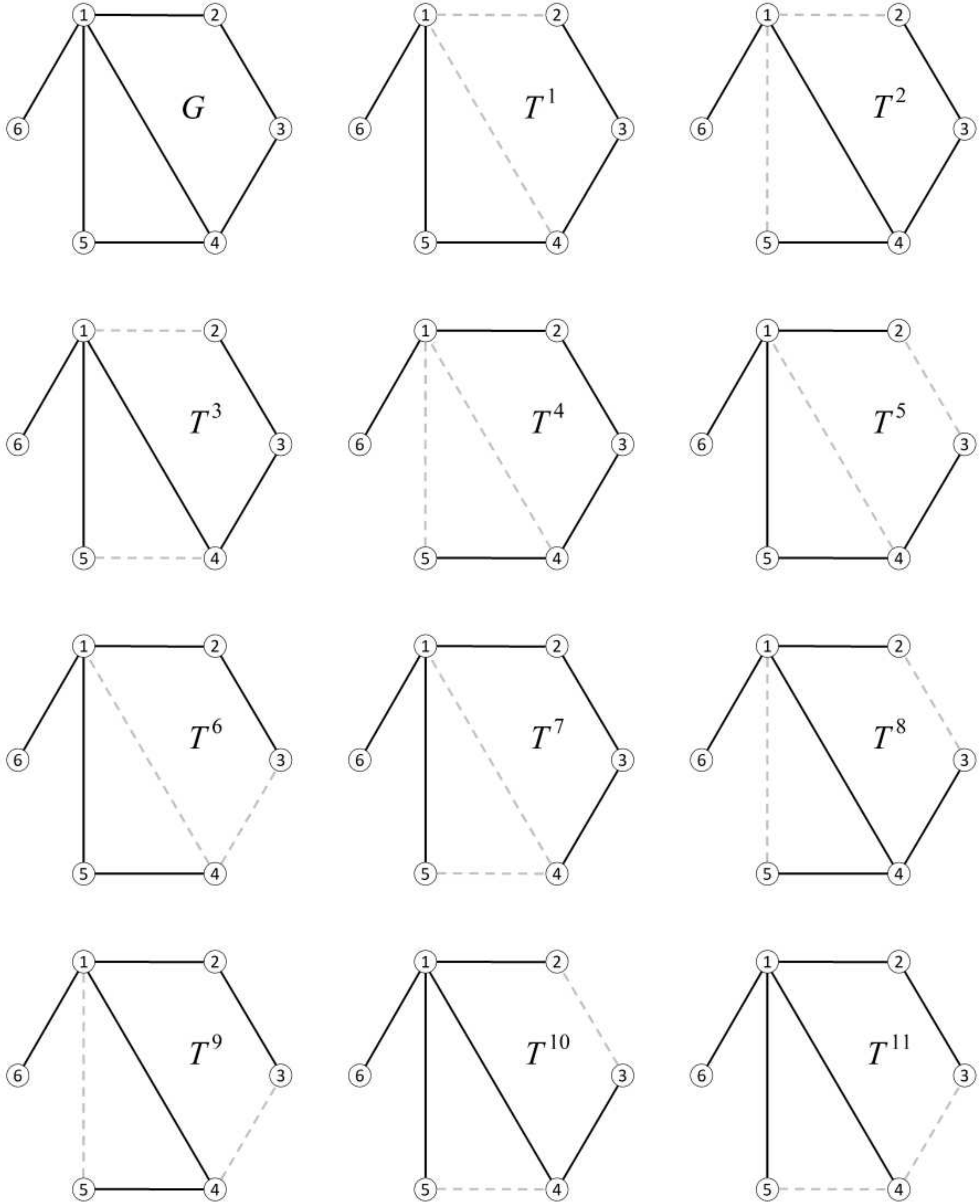
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Multi Criteria Decision Making

Analytic Hierarchy Process (Saaty, 1977)

Criterion tree

Pairwise comparison matrix

# The aim of multiple criteria decision analysis

The aim is **to select the overall best one** from a finite set of *alternatives*, with respect to a finite set of **attributes (criteria)**,
or,
**to rank** the alternatives,
or,
**to classify** the alternatives.

# Properties of multiple criteria decision problems

- criteria contradict each other
- there is not a single best solution, that is optimal with respect to each criterion
- subjective factors influence the decision
- contradictive individual opinions have to be aggregated

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## Examples of multi criteria decision problems

- tenders, public procurements, privatizations
- evaluation of applications
- environmental studies
- ranking, classification

# Main tasks in multi criteria decision problems

- to assign weights of importance to the criteria
- to evaluate the alternatives
- to aggregate the evaluations with the weights of criteria
- sensitivity analysis

# Decomposition of the goal: tree of criteria

- main criterion 1
  - criterion 1.1
  - criterion 1.2
  - criterion 1.3
  - criterion 1.4
  - criterion 1.5
- main criterion 2
  - criterion 2.1
  - criterion 2.2
- main criterion 3
  - criterion 3.1
    - subcriterion 3.1.1
    - subcriterion 3.1.2
  - criterion 3.2

# Estimating weights from pairwise comparisons

'How many times criterion 1 is more important than criterion 2?'

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix},$$

is given, where for any $i, j = 1, \dots, n$ indices

$$a_{ij} > 0, \qquad a_{ij} = \frac{1}{a_{ji}}.$$

The aim is to find the $\mathbf{w} = (w_1, w_2, \dots, w_n)^\top \in \mathbb{R}^n_+$ weight vector such that ratios $\frac{w_i}{w_j}$ are *close enough* to $a_{ij}$s.

# Evaluation of the alternatives

Alternatives are evaluated directly, or by using a function, or by pairwise comparisons as before.

'How many times alternative 1 is better than alternative 2 with respect to criterion 1.1?'

$$\mathbf{B} = \begin{pmatrix} 1 & b_{12} & b_{13} & \dots & b_{1m} \\ b_{21} & 1 & b_{23} & \dots & b_{2m} \\ b_{31} & b_{32} & 1 & \dots & b_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \dots & 1 \end{pmatrix}$$

# Aggregation of the evaluations

- total scores are calculated as a weighted sum of the evaluations with respect to leaf nodes of the criteria tree (bottom up);
- partial sums are informative

# Weighting methods

Eigenvector Method (Saaty): $\mathbf{A}\mathbf{w} = \lambda_{max}\mathbf{w}$.

Logarithmic Least Squares Method (LLSM):

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \log a_{ij} - \log \frac{w_i}{w_j} \right)^2$$

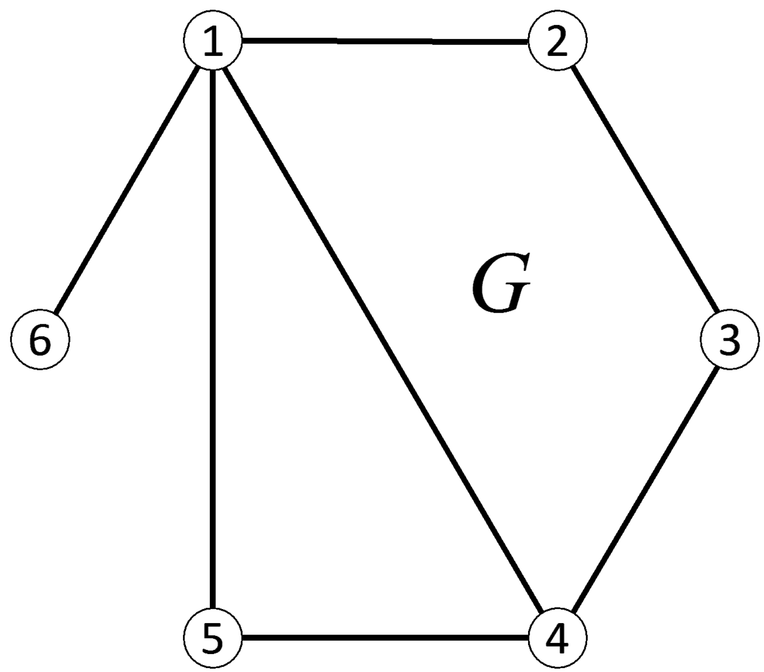
$$\sum_{i=1}^{n} w_i = 1, \qquad w_i > 0, \quad i = 1, 2, \ldots, n.$$

# incomplete pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$

# incomplete pairwise comparison matrix and its graph

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



# The Logarithmic Least Squares (LLS) problem

$$\min \sum_{\substack{i,j\,: \\ a_{ij} \text{ is known}}} \left[\log a_{ij} - \log\left(\frac{w_i}{w_j}\right)\right]^2$$

$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

The most common normalizations are $\sum_{i=1}^{n} w_i = 1$, $\prod_{i=1}^{n} w_i = 1$ and $w_1 = 1$.

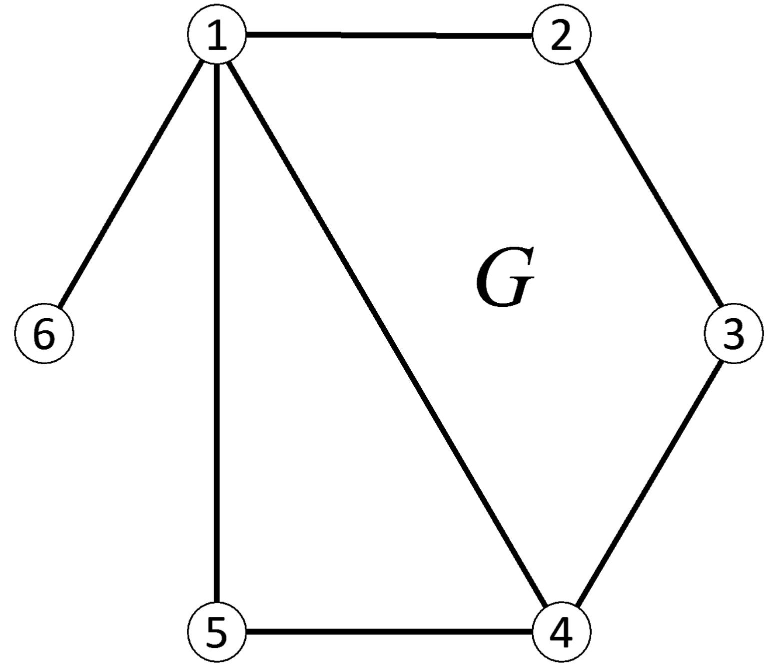
**Theorem** (Bozóki, Fülöp, Rónyai, 2010): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph $G$ is connected. Then the optimal solution $\mathbf{w} = \exp \mathbf{y}$ of the logarithmic least squares problem is the unique solution of the following system of linear equations:

$$
\begin{aligned}
(\mathbf{L}\mathbf{y})_i &= \sum_{k:e(i,k)\in E(G)} \log a_{ik} \qquad \text{for all } i = 1, 2, \ldots, n, \\
y_1 &= 0
\end{aligned}
$$

where $\mathbf{L}$ denotes the Laplacian matrix of $G$ ($\ell_{ii}$ is the degree of node $i$ and $\ell_{ij} = -1$ if nodes $i$ and $j$ are adjacent).

## example

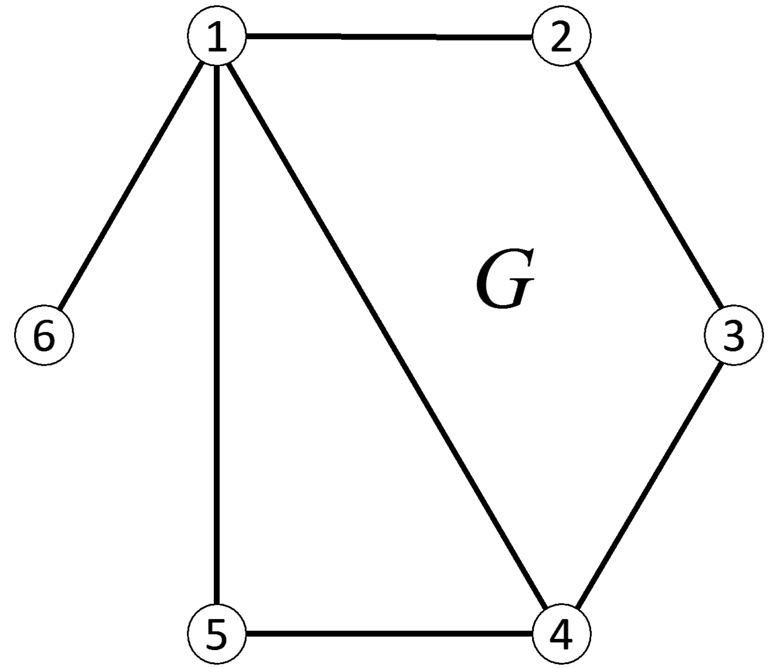
$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$




$$\begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(=0) \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} \log(a_{12}\, a_{14}\, a_{15}\, a_{16}) \\ \log(a_{21}\, a_{23}) \\ \log(a_{32}\, a_{34}) \\ \log(a_{41}\, a_{43}\, a_{45}) \\ \log(a_{51}\, a_{54}) \\ \log a_{61} \end{pmatrix}$$

# The spanning tree approach (Tsyganok, 2000, 2010)

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



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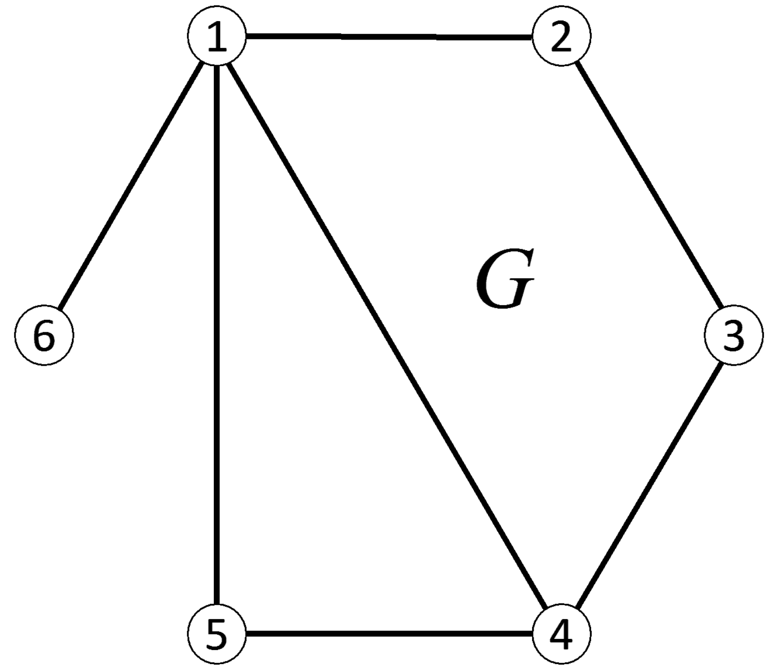


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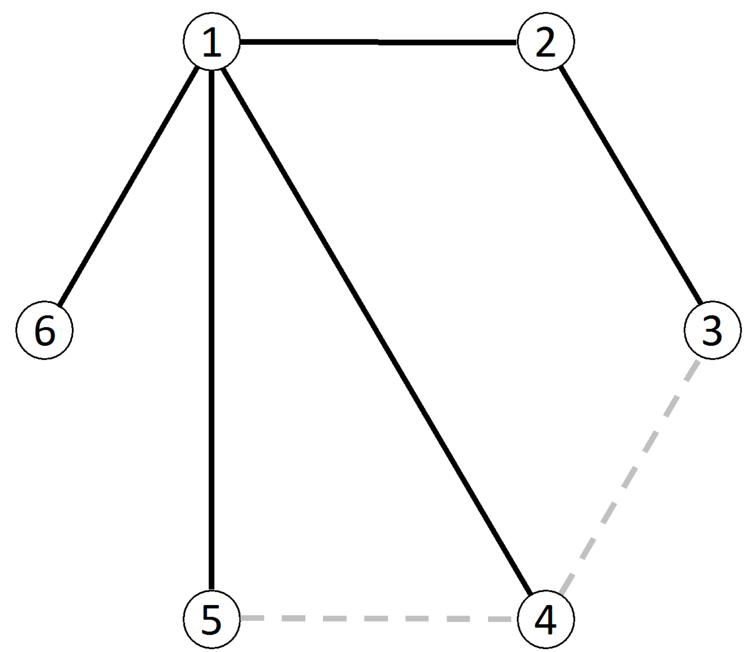

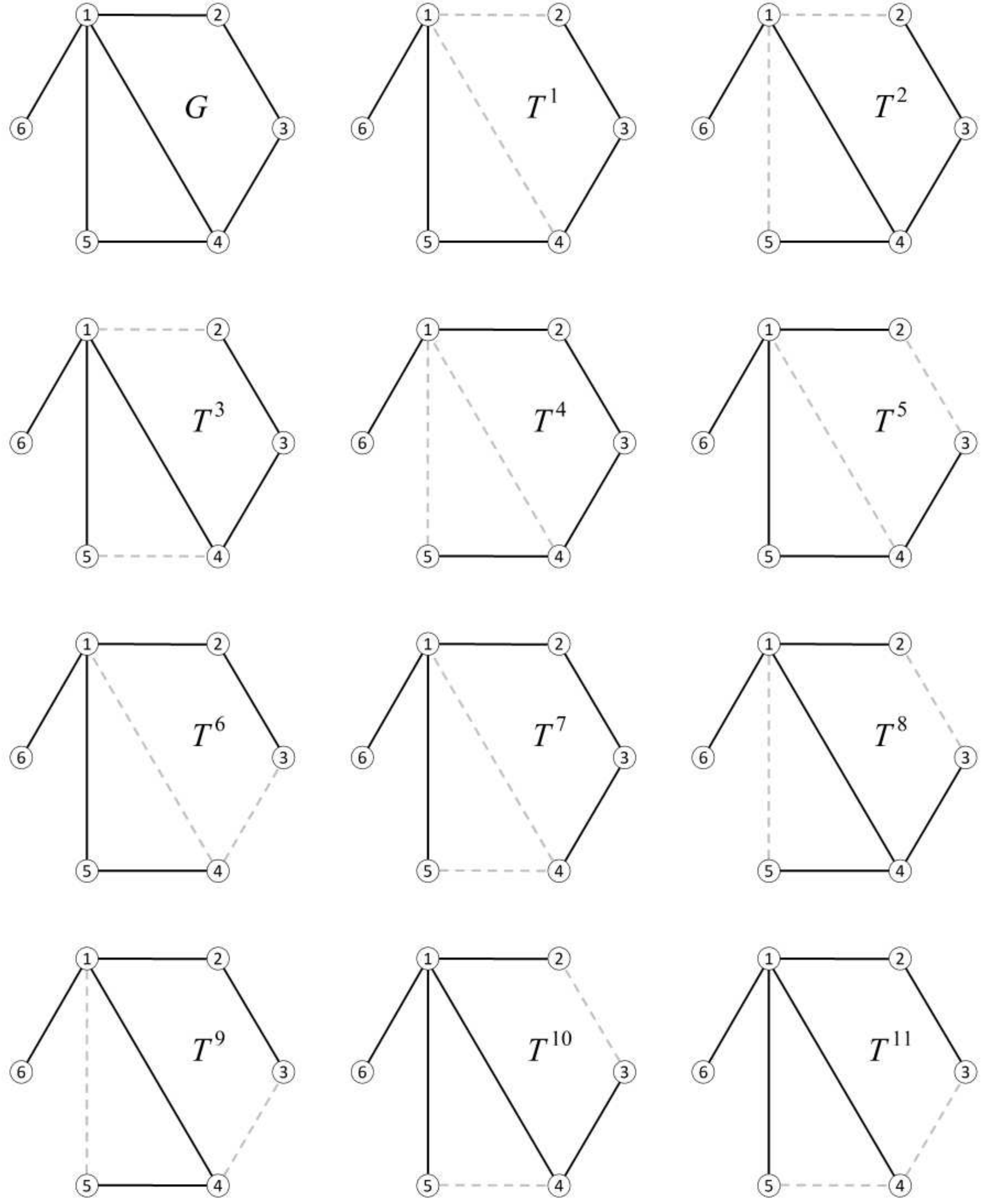
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## The spanning tree approach

Every spanning tree induces a weight vector.

Natural ways of aggregation: arithmetic mean, geometric mean etc.

**Theorem** (Lundy, Siraj, Greco, 2017): The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete pairwise comparison matrices.

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**Theorem** (Bozóki, Tsyganok): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph is connected. Then the optimal solution of the logarithmic least squares problem is equal, up to a scalar multiplier, to the geometric mean of weight vectors calculated from all spanning trees.

## proof

Let $G$ be the connected graph associated to the (in)complete pairwise comparison matrix $\mathbf{A}$ and let $E(G)$ denote the set of edges. The edge between nodes $i$ and $j$ is denoted by $e(i,j)$.

The Laplacian matrix of graph $G$ is denoted by $\mathbf{L}$. Let $T^1, T^2, \ldots, T^s, \ldots, T^S$ denote the spanning trees of $G$, where $S$ denotes the number of spanning trees. $E(T^s)$ denotes the set of edges in $T^s$.

Let $\mathbf{w}^s, s = 1, 2, \ldots, S$, denote the weight vector calculated from spanning tree $T^s$. Weight vector $\mathbf{w}^s$ is unique up to a scalar multiplication. Assume without loss of generality that $w_1^s = 1$.

Let $\mathbf{y}^s := \log \mathbf{w}^s,\ s = 1, 2, \ldots, S$, where the logarithm is taken element-wise.

## proof

Let $\mathbf{w}^{LLS}$ denote the optimal solution to the incomplete Logarithmic Least Squares problem (normalized by $w_1^{LLS} = 1$) and $\mathbf{y}^{LLS} := \log \mathbf{w}^{LLS}$, then

$$\left(\mathbf{L}\mathbf{y}^{LLS}\right)_i = \sum_{k:e(i,k)\in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \dots, n,$$

where $b_{ik} = \log a_{ik}$ for all $(i,k) \in E(G)$.

$b_{ik} = -b_{ki}$ for all $(i,k) \in E(G)$.

In order to prove the theorem, it is sufficient to show that

$$\left(\mathbf{L}\frac{1}{S}\sum_{s=1}^{S}\mathbf{y}^{s}\right)_i = \sum_{k:e(i,k)\in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \dots, n.$$

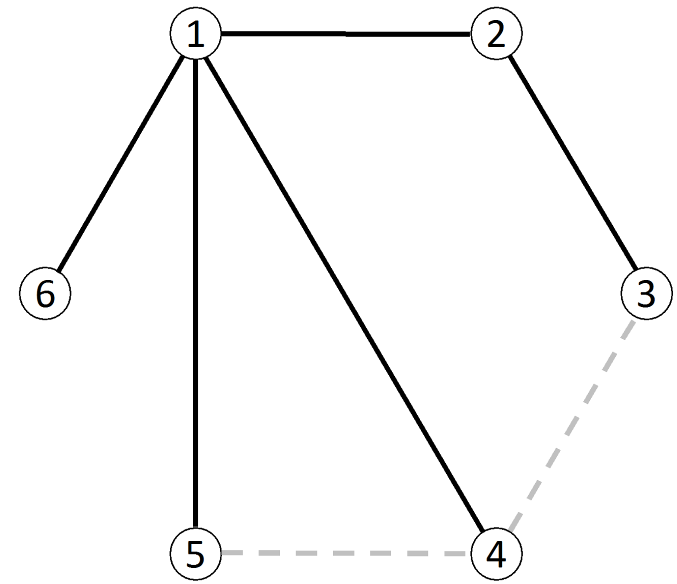
## proof

Challenge: the Laplacian matrices of the spanning trees are different from the Laplacian of $G$.

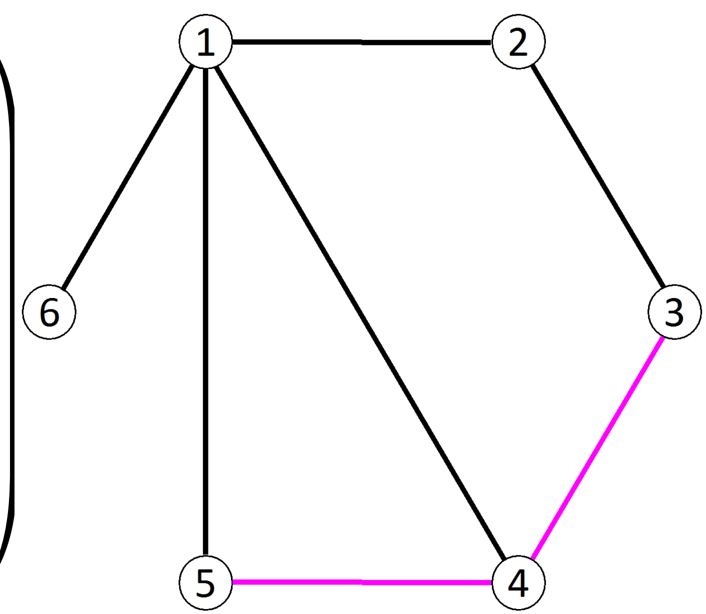
Consider an arbitrary spanning tree $T^s$. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$.
Introduce the incomplete pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G) \backslash E(T^s)$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.
Note that the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$).

# proof

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & & & \\ a_{41} & & & 1 & & \\ a_{51} & & & & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$




$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{32}a_{21}a_{14} & & \\ a_{41} & & a_{41}a_{12}a_{23} & 1 & a_{41}a_{15} & \\ a_{51} & & & a_{51}a_{14} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



## proof

Consider an arbitrary spanning tree $T^s$. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$. Introduce the incomplete pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G) \backslash E(T^s)$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.

Note that the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$).

Since weight vector $\mathbf{w}^s$ is generated by the matrix elements belonging to spanning tree $T^s$, it is the optimal solution of the $LLS$ problem regarding $\mathbf{A}^s$, too. Equivalently, the following system of linear equations holds.

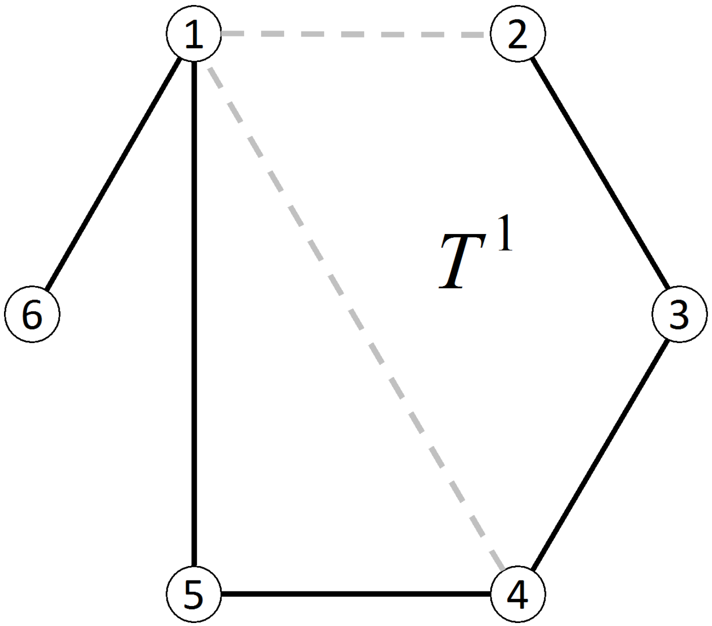
$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k) \in E(T^s)} b_{ik} + \sum_{k:e(i,k) \in E(G) \backslash E(T^s)} b_{ik}^s \quad \text{for all } i = 1, \ldots, n$$

## proof

### Lemma

$$\sum_{s=1}^{S} \left( \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^{s} \right) = S \sum_{k:e(i,k)\in E(G)} b_{ik}$$

# proof of the lemma



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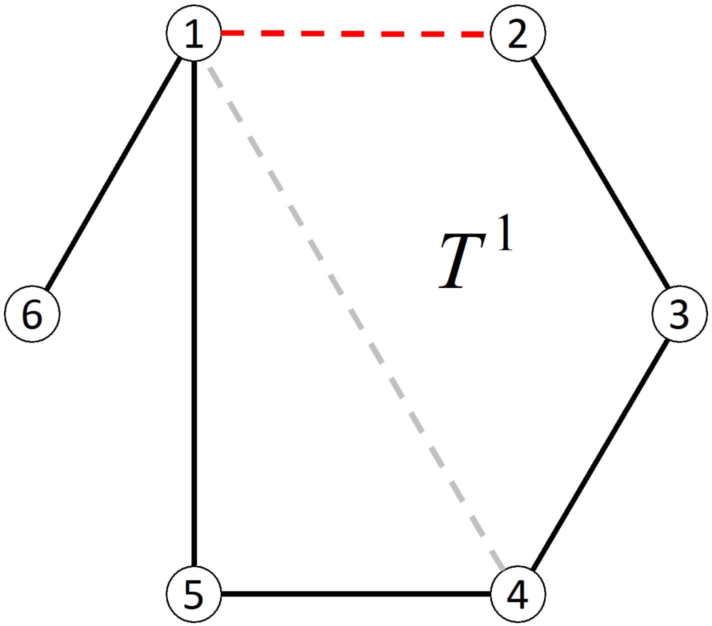

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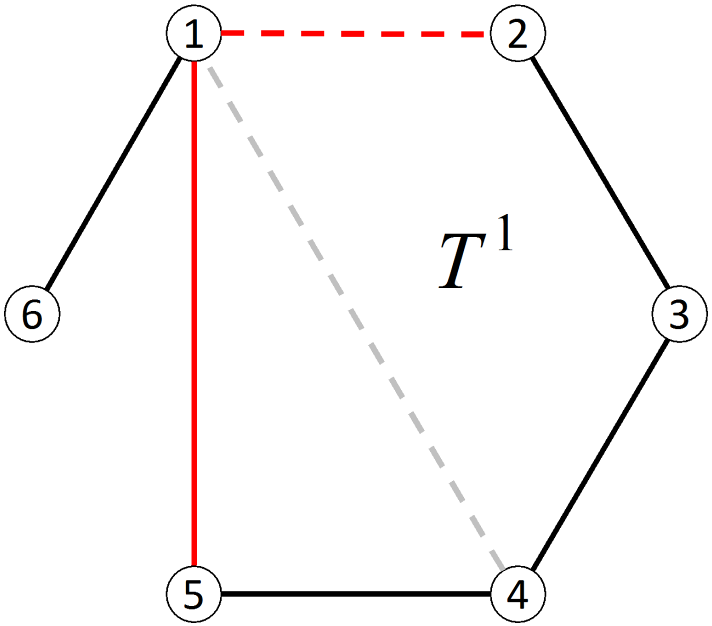

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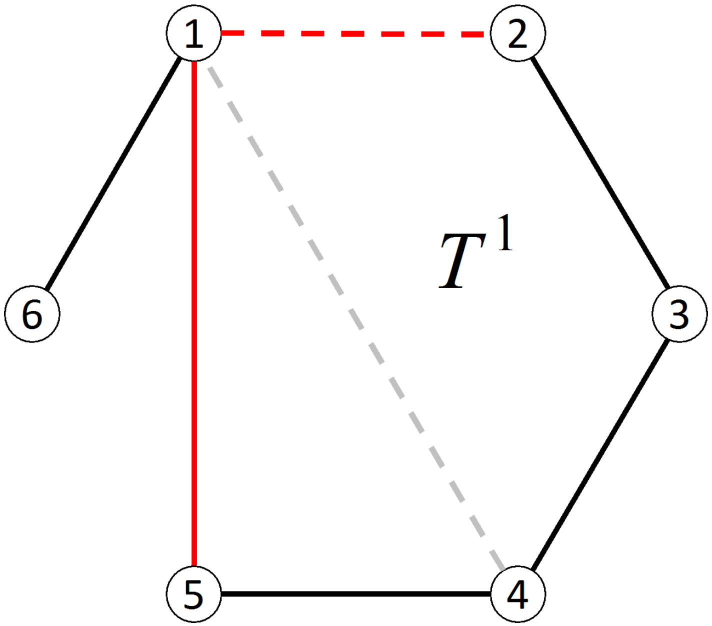
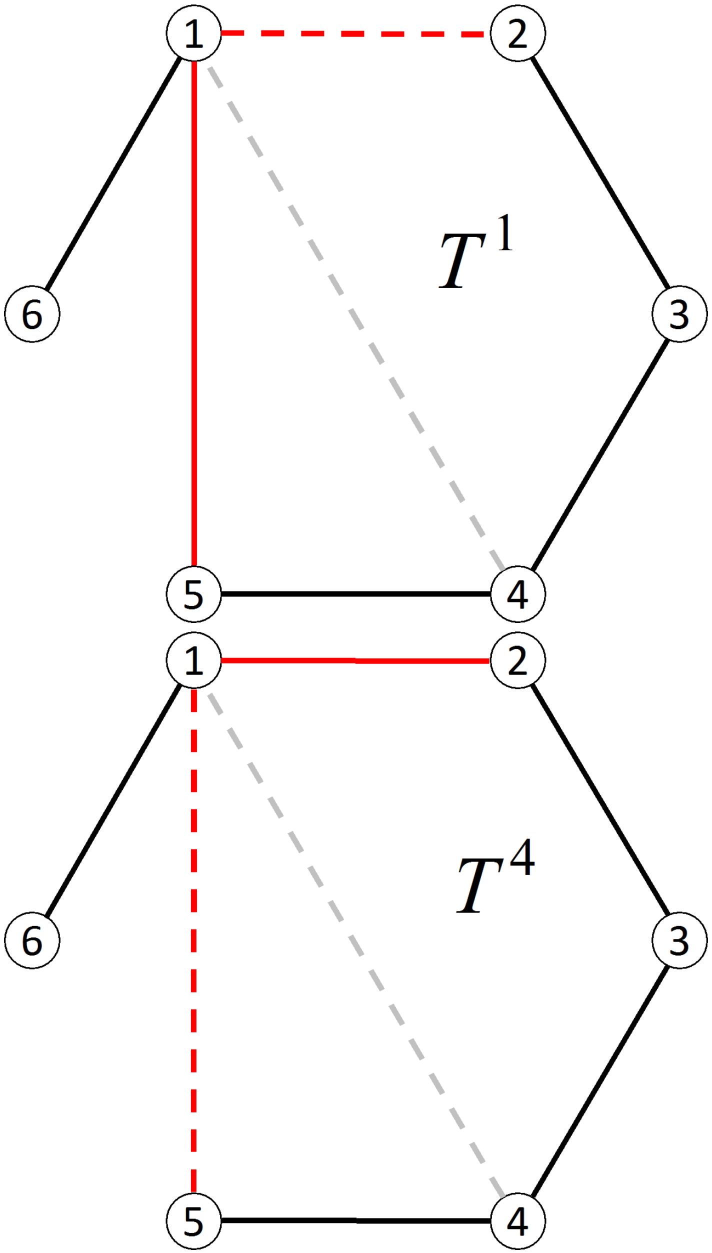


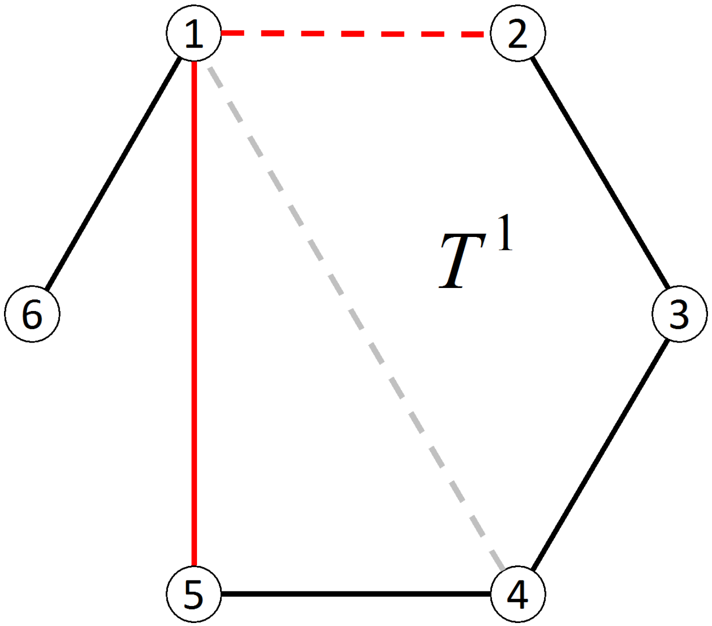

$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

# proof of the lemma




$$b_{12}^{1} = b_{15} + b_{54} + b_{43} + b_{32}$$

# proof of the lemma

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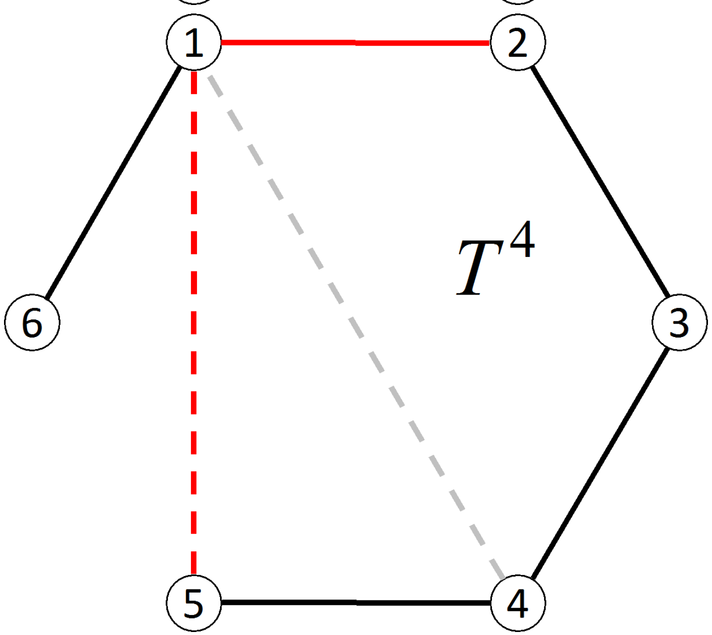
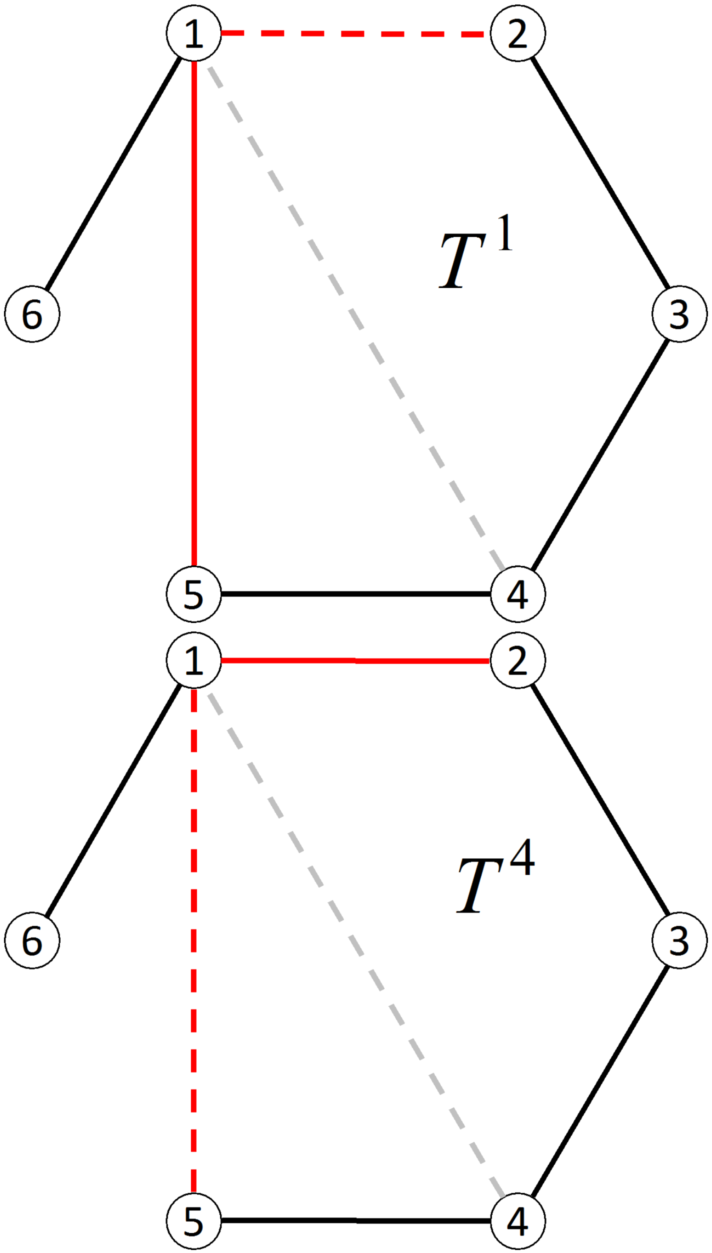



$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

# proof of the lemma




$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

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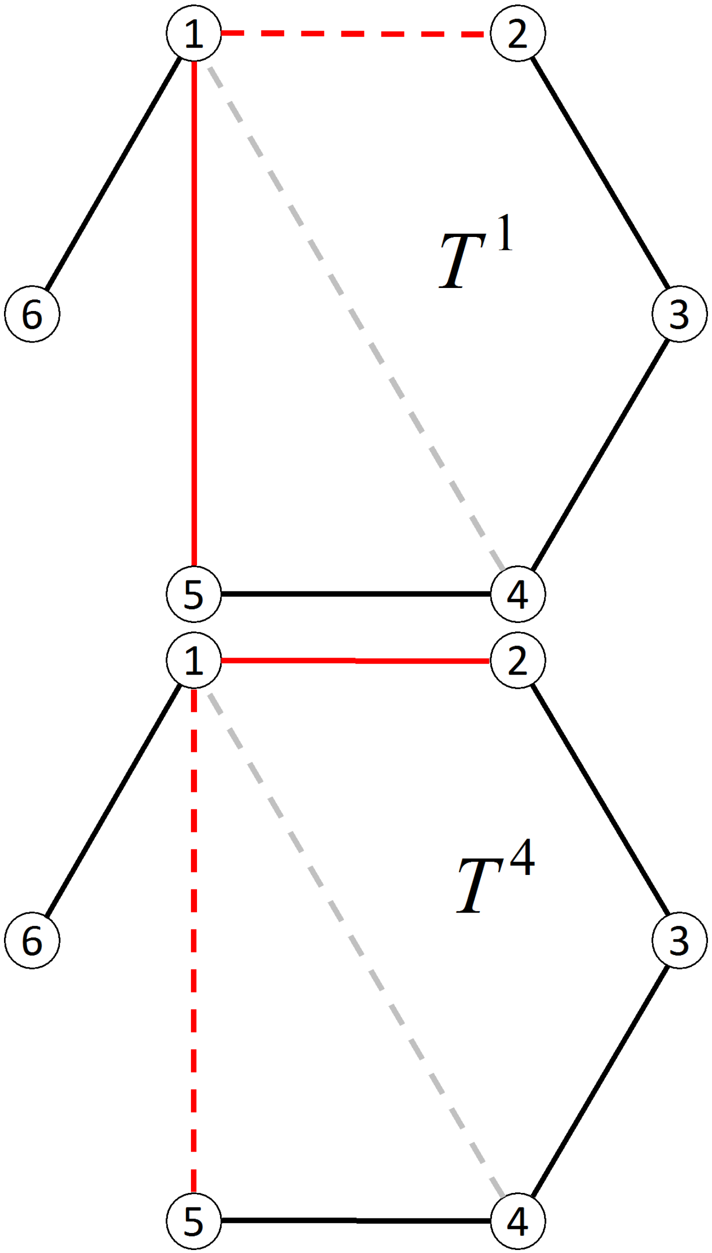


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$$b^1_{12} + b^4_{15} = b_{12} + b_{15}$$

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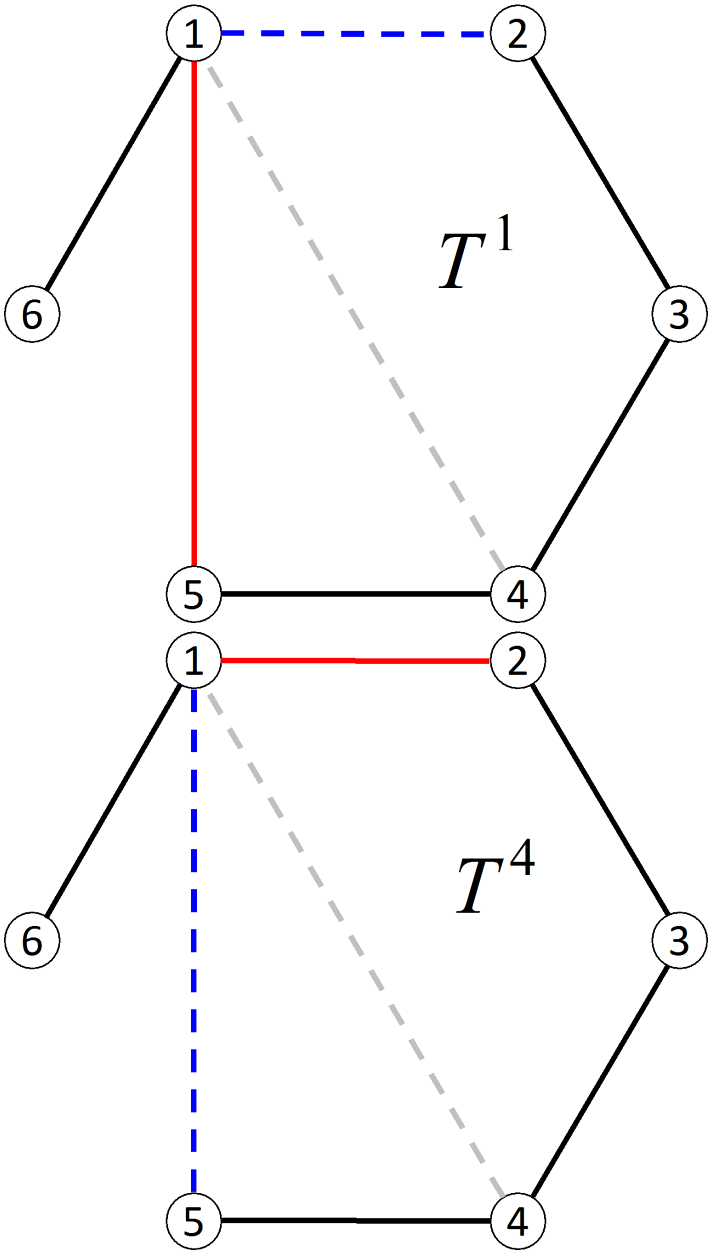


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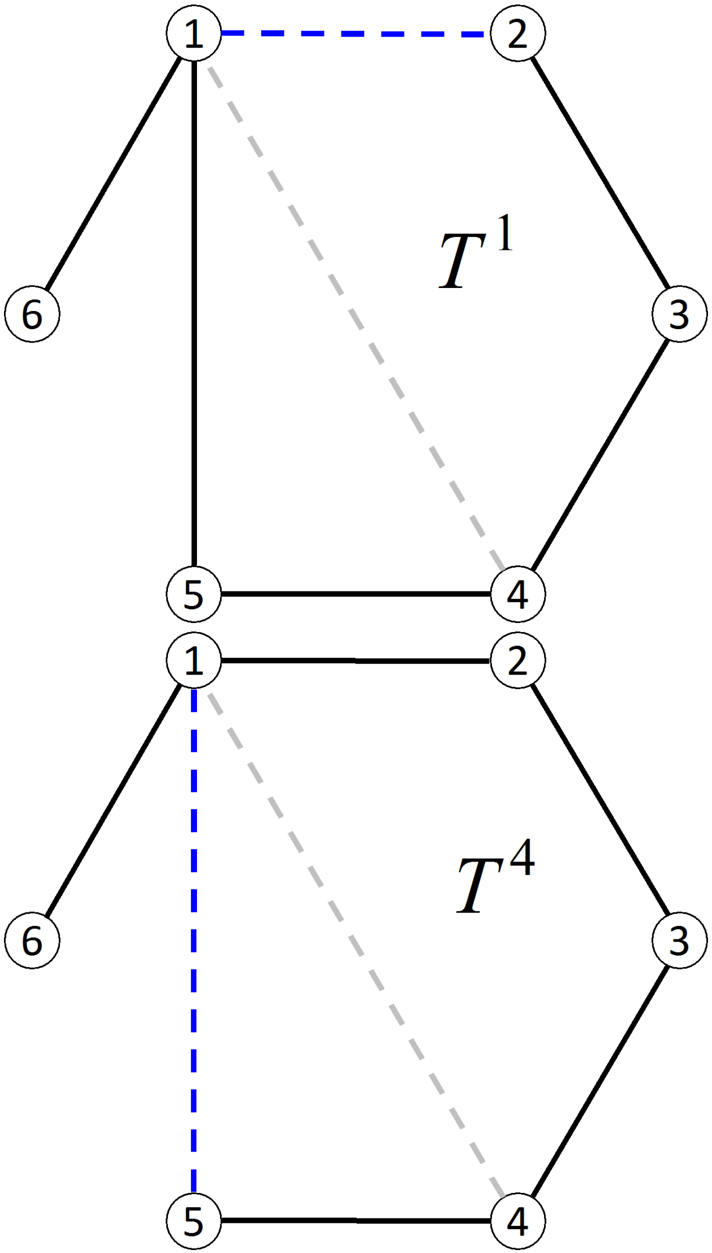


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$$b^1_{12} + b^4_{15} = b_{12} + b_{15}$$

## proof

Finally, to complete the proof, take the sum of equations

$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^s \quad \text{for all } i = 1, \ldots, n$$

for all $s = 1, 2, \ldots, S$ and apply the lemma

$$\sum_{s=1}^{S} \left( \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^s \right) = S \sum_{k:e(i,k)\in E(G)} b_{ik}$$

to conclude that $\mathbf{y}^{LLS} = \frac{1}{S}\sum_{s=1}^{S} \mathbf{y}^s$. $\square$

**Remark.** Complete pairwise comparison matrices ($S = n^{n-2}$) are included in our theorem as a special case, and our proof can also be considered as a second, and shorter proof of the theorem of Lundy, Siraj and Greco (2017).

# Conclusions

The equivalence of two fundamental weighting methods has been shown.

The advantages of two approaches have been united.

## Main references in historical order 1/2

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Thank you for attention.

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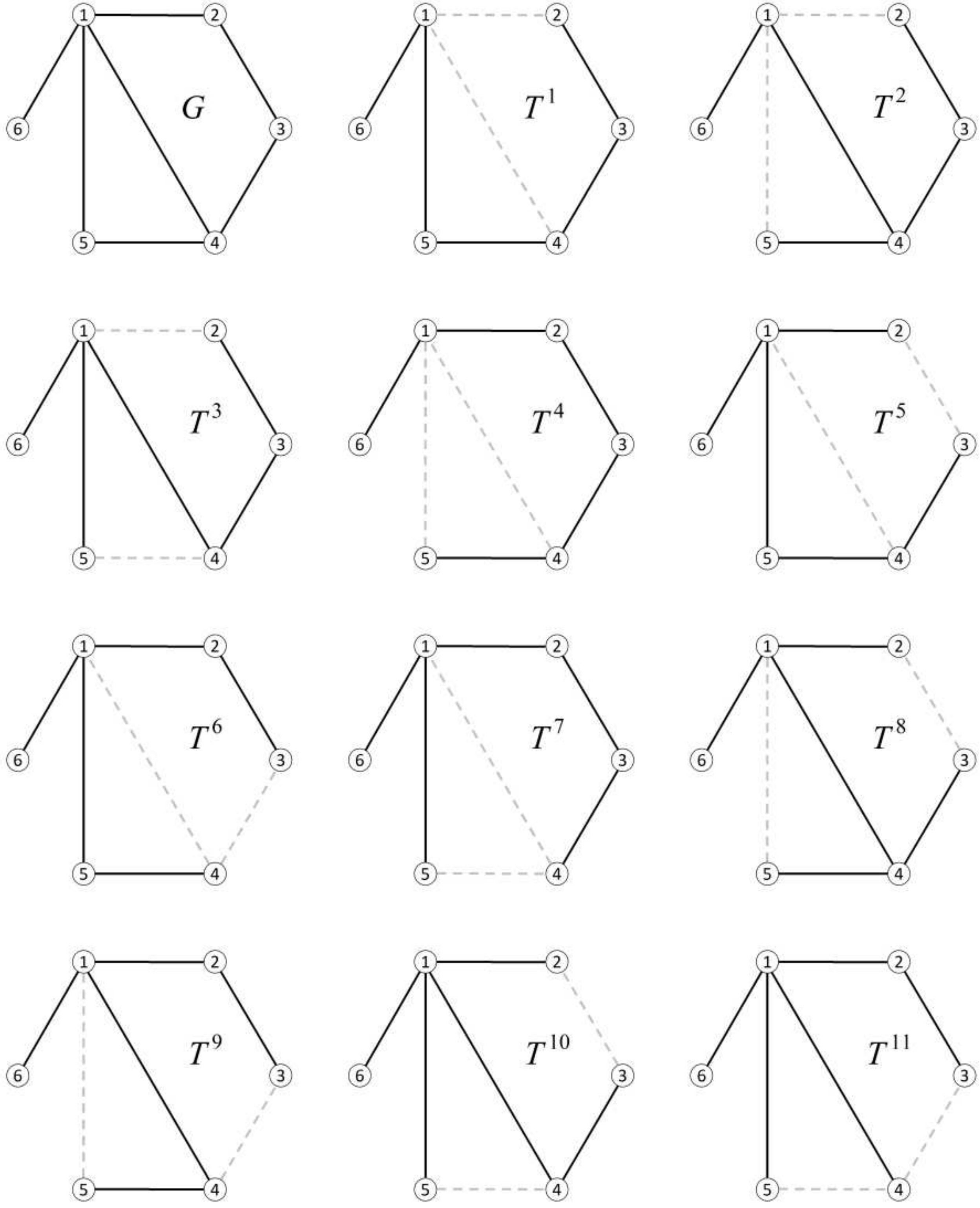
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