Appendix to the paper
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Let $\mathbf{D} = \mathrm{diag}(1, 1/x_1, \dots, 1/x_{n-1})$, and let $\mathbf{e} = (1, \dots, 1)^T$. For $\mathbf{A} \in \mathcal{PCM}_n$,

$$\mathbf{e}\mathbf{e}^T - \mathbf{U}_i\mathbf{V}_i^T = \mathbf{D}^{-1}\mathbf{A}\mathbf{D} \tag{6}$$

holds for $i = 1, 2$. For $i = 1$, $\mathbf{A}$ has form (3) (Case 1) and

$$\mathbf{U}_1 = \begin{pmatrix} 0 & 1 \\ 1-1/\delta & 0 \\ 1-1/\gamma & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times 2}, \qquad \mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-\delta \\ 0 & 1-\gamma \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times 2}. \tag{7}$$

For $i = 2$, $\mathbf{A}$ has form (4) or (5) (Case 2) and

$$\mathbf{U}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-1/\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1-1/\gamma & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times 4}, \mathbf{V}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\gamma \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times 4}. \tag{8}$$

Lemma 1. (Matrix determinant lemma (Harville, 2008, Theorem 18.1.1)) *If* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *is invertible, and* $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n\times m}$*, then*

$$\det(\mathbf{A} + \mathbf{U}\mathbf{V}^T) = \det(\mathbf{I}_m + \mathbf{V}^T\mathbf{A}^{-1}\mathbf{U})\det(\mathbf{A}),$$

where $\mathbf{I}_m$ *denotes the identity matrix of size* $m \times m$*.*

Lemma 2. (Sherman–Morrison formula (Sherman and Morrison, 1950)) *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$*,* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*. If* $\mathbf{A}$ *is invertible and* $1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u} \neq 0$*, then* $\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)^{-1}$ *exists, and*

$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{1}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}.$$

Let $\mathbf{A} \in \mathcal{PCM}_n$ be a double perturbed PCM and $\mathbf{U}_i, \mathbf{V}_i$ be as in (6). Let the matrix $\mathbf{K}_\mathbf{A}(\lambda) \in \mathbb{R}^{n\times n}$ be defined as follows:

$$\mathbf{K}_\mathbf{A}(\lambda) = \lambda\mathbf{I} + \mathbf{U}_i\mathbf{V}_i^T - \mathbf{e}\mathbf{e}^T = \lambda\mathbf{I} - \mathbf{D}^{-1}\mathbf{A}\mathbf{D},$$

where $\mathbf{I}$ denotes $\mathbf{I}_n$, $\mathbf{e} = (1, \dots, 1)^T \in \mathbb{R}^n$, $i = 1$ in Case 1, $i = 2$ in Case 2, and the second equation follows from (6).

Lemma 3. *The characteristic polynomial of the double perturbed PCM* $\mathbf{A} \in \mathcal{PCM}_n$ *is*

$$p_{\mathbf{A}}(\lambda) = (-1)^n \det(\mathbf{K}_{\mathbf{A}}(\lambda)).$$

Proof. As before, $i = 1$ in Case 1 and $i = 2$ in Case 2.

$$\begin{aligned}
p_{\mathbf{A}}(\lambda) &= \det(\mathbf{A} - \lambda\mathbf{I}) \\
&= (-1)^n \det(\lambda\mathbf{I} - \mathbf{A}) \\
&= (-1)^n \det\left(\lambda\mathbf{I} + \mathbf{D}\left(\mathbf{U}_i\mathbf{V}_i^T - \mathbf{e}\mathbf{e}^T\right)\mathbf{D}^{-1}\right) \\
&= (-1)^n \det\left(\mathbf{D}\left(\lambda\mathbf{I} + \mathbf{U}_i\mathbf{V}_i^T - \mathbf{e}\mathbf{e}^T\right)\mathbf{D}^{-1}\right) \\
&= (-1)^n \det(\mathbf{D}) \det\left(\lambda\mathbf{I} + \mathbf{U}_i\mathbf{V}_i^T - \mathbf{e}\mathbf{e}^T\right) \det\left(\mathbf{D}^{-1}\right) \\
&= (-1)^n \det(\mathbf{K}_{\mathbf{A}}(\lambda)).
\end{aligned}$$

□

Lemma 4. $\det(\lambda\mathbf{I} - \mathbf{e}\mathbf{e}^T) = \lambda^n - n\lambda^{n-1}$.

Proof. If $\lambda = 0$, then both sides of the equation are 0. If $\lambda \neq 0$, apply Lemma 1 with $m = 1$, $\mathbf{A} = \lambda\mathbf{I}$, $\mathbf{U} = -\mathbf{e}$, $\mathbf{V} = \mathbf{e}$:

$$\det(\lambda\mathbf{I} - \mathbf{e}\mathbf{e}^T) = (1 - \mathbf{e}^T(\lambda\mathbf{I})^{-1}\mathbf{e}) \det(\lambda\mathbf{I}) = \lambda^n - n\lambda^{n-1}.$$

□

Lemma 5. *If* $\lambda \neq 0$ *and* $\lambda \neq n$, *then* $\left(\lambda\mathbf{I} - \mathbf{e}\mathbf{e}^T\right)^{-1}$ *exists, and*

$$\left(\lambda\mathbf{I} - \mathbf{e}\mathbf{e}^T\right)^{-1} = \frac{1}{\lambda\,(\lambda - n)}\mathbf{e}\mathbf{e}^T + \frac{1}{\lambda}\mathbf{I}.$$

Proof. Apply the Sherman–Morrison formula (Lemma 2) with $\mathbf{A} = \lambda\mathbf{I}$, $\mathbf{u} = -\mathbf{e}$, $\mathbf{v} = \mathbf{e}$. □

Lemma 6. *Let* $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n\times m}$ *be arbitrary matrices. If* $\lambda \neq 0$ *and* $\lambda \neq n$, *then*

$$\det\left(\lambda\mathbf{I}_n + \mathbf{U}\mathbf{V}^T - \mathbf{e}\mathbf{e}^T\right) = \left(\lambda^n - n\lambda^{n-1}\right) \det\left(\mathbf{I}_m + \frac{1}{\lambda(\lambda - n)}\mathbf{V}^T\mathbf{e}\mathbf{e}^T\mathbf{U} + \frac{1}{\lambda}\mathbf{V}^T\mathbf{U}\right).$$

Proof. Apply Lemma 1 with $\mathbf{A} = \lambda\mathbf{I}_n - \mathbf{e}\mathbf{e}^T$. According to Lemma 5, $\mathbf{A}$ is invertible. Utilizing Lemmas 1, 4 and 5 the following equations hold:

$$\begin{aligned}
&\det\left(\left(\lambda\mathbf{I}_n - \mathbf{e}\mathbf{e}^T\right) + \mathbf{U}\mathbf{V}^T\right) \\
&= \det\left(\mathbf{I}_m + \mathbf{V}^T\left(\lambda\mathbf{I}_n - \mathbf{e}\mathbf{e}^T\right)^{-1}\mathbf{U}\right) \det\left(\lambda\mathbf{I}_n - \mathbf{e}\mathbf{e}^T\right) \\
&= \det\left(\mathbf{I}_m + \frac{1}{\lambda(\lambda - n)}\mathbf{V}^T\mathbf{e}\mathbf{e}^T\mathbf{U} + \frac{1}{\lambda}\mathbf{V}^T\mathbf{U}\right)\left(\lambda^n - n\lambda^{n-1}\right).
\end{aligned}$$

□

We can write the characteristic polynomial of double perturbed PCMs in explicit form.

Proposition 1. *Let $n \geq 4$. The characteristic polynomial of a double perturbed PCM in form (3) (Case 1) is*

$$p_{\mathbf{P}}(\lambda) = (-1)^n \lambda^{n-3} \left(\lambda^3 - n\lambda^2 - \left(\frac{\gamma}{\delta} + \frac{\delta}{\gamma}\right) - (n-3)\left(\gamma + \delta + \frac{1}{\gamma} + \frac{1}{\delta}\right) + 4n - 10 \right).$$

Proof. Lemma 3 implies that

$$p_{\mathbf{P}}(\lambda) = (-1)^n \det(\mathbf{K}_{\mathbf{P}}(\lambda)) = (-1)^n \det\left(\lambda \mathbf{I} + \mathbf{U}_1 {\mathbf{V}_1}^T - \mathbf{e}\mathbf{e}^T\right),$$

where $\mathbf{U}_1$ and $\mathbf{V}_1$ are defined by (7). Suppose that $\lambda \neq n$ and $\lambda \neq 0$. According to Lemma 6

$$\begin{aligned} p_{\mathbf{P}}(\lambda) &= (-1)^n \left(\lambda^n - n\lambda^{n-1}\right) \det\left(\mathbf{I}_2 + \frac{1}{\lambda(\lambda - n)} {\mathbf{V}_1}^T \mathbf{e}\mathbf{e}^T \mathbf{U}_1 + \frac{1}{\lambda} {\mathbf{V}_1}^T \mathbf{U}_1\right) \\ &= (-1)^n \left(\lambda^n - n\lambda^{n-1}\right) \det(\mathbf{S}) \\ &= (-1)^n \lambda^{n-3} \left(\lambda^3 - n\lambda^2 - \left(\frac{\gamma}{\delta} + \frac{\delta}{\gamma}\right) - (n-3)\left(\gamma + \delta + \frac{1}{\gamma} + \frac{1}{\delta}\right) + 4n - 10 \right), \end{aligned}$$

where

$$\mathbf{S} = \begin{pmatrix} 1 + \frac{2-1/\delta-1/\gamma}{\lambda(\lambda-n)} & \frac{1}{\lambda(\lambda-n)} + \frac{1}{\lambda} \\ \frac{(2-\delta-\gamma)(1-1/\delta)+(2-\delta-\gamma)(1-1/\gamma)}{\lambda(\lambda-n)} + \frac{(1-\delta)(1-1/\delta)}{\lambda} + \frac{(1-\gamma)(1-1/\gamma)}{\lambda} & 1 + \frac{2-\delta-\gamma}{\lambda(\lambda-n)} \end{pmatrix}.$$

A polynomial of degree n is uniquely determined by $n+1$ points, and we have calculated $p_{\mathbf{P}}(\lambda)$ in all but two points, which completes the proof. □

Proposition 2. *Let $n \geq 4$. The characteristic polynomial of a double perturbed PCM in form (5) (Case 2B) is*

$$p_{\mathbf{R}}(\lambda) = (-1)^n \lambda^{n-5} \left(\lambda^5 - n\lambda^4 - (n-2)\left(\gamma + \delta + \frac{1}{\gamma} + \frac{1}{\delta} - 4\right)\lambda^2 - c\lambda - (n-4)c \right),$$

where

$$c = \frac{(\gamma-1)^2(\delta-1)^2}{\gamma\delta}.$$

Furthermore, the characteristic polynomial of a double perturbed PCM in form (4) (Case 2A), $p_{\mathbf{Q}}(\lambda)$ is a special case of $p_{\mathbf{R}}(\lambda)$ with $n = 4$. Namely,

$$p_{\mathbf{Q}}(\lambda) = \lambda^4 - 4\lambda^3 - 2\left(\gamma + \delta + \frac{1}{\gamma} + \frac{1}{\delta} - 4\right)\lambda - \frac{(\gamma-1)^2(\delta-1)^2}{\gamma\delta}.$$

Proof. Lemma 3 implies that

$$p_{\mathbf{R}}(\lambda) = (-1)^n \det(\mathbf{K}_{\mathbf{R}}(\lambda)) = (-1)^n \det\left(\lambda \mathbf{I} + \mathbf{U}_2 {\mathbf{V}_2}^T - \mathbf{e}\mathbf{e}^T\right),$$

where $\mathbf{U}_2$ and $\mathbf{V}_2$ are defined by (8). Suppose that $\lambda \neq n$ and $\lambda \neq 0$. According to Lemma 6

$$p_{\mathbf{R}}(\lambda) = (-1)^n \left(\lambda^n - n\lambda^{n-1}\right) \det\left(\mathbf{I}_4 + \frac{1}{\lambda(\lambda - n)} {\mathbf{V}_2}^T \mathbf{e}\mathbf{e}^T \mathbf{U}_2 + \frac{1}{\lambda} {\mathbf{V}_2}^T \mathbf{U}_2\right)$$

$$= (-1)^n \left(\lambda^n - n\lambda^{n-1}\right) \det(\mathbf{T})$$
$$= (-1)^n \lambda^{n-5} \left(\lambda^5 - n\lambda^4 - (n-2) \left(\gamma + \delta + \frac{1}{\gamma} + \frac{1}{\delta} - 4 \right) \lambda^2 - c\lambda - (n-4)c \right),$$

where

$$\mathbf{T} = \begin{pmatrix} 1 + \frac{1-1/\delta}{\lambda(\lambda-n)} & \frac{1}{\lambda(\lambda-n)} + \frac{1}{\lambda} & \frac{1-1/\gamma}{\lambda(\lambda-n)} & \frac{1}{\lambda(\lambda-n)} \\ \frac{(1-\delta)(1-1/\delta)}{\lambda(\lambda-n)} + \frac{(1-\delta)(1-1/\delta)}{\lambda} & 1 + \frac{1-\delta}{\lambda(\lambda-n)} & \frac{(1-\delta)(1-1/\gamma)}{\lambda(\lambda-n)} & \frac{1-\delta}{\lambda(\lambda-n)} \\ \frac{1-1/\delta}{\lambda(\lambda-n)} & \frac{1}{\lambda(\lambda-n)} & 1 + \frac{1-1/\gamma}{\lambda(\lambda-n)} & \frac{1}{\lambda(\lambda-n)} + \frac{1}{\lambda} \\ \frac{(1-\gamma)(1-1/\delta)}{\lambda(\lambda-n)} & \frac{1-\gamma}{\lambda(\lambda-n)} & \frac{(1-\gamma)(1-1/\gamma)}{\lambda(\lambda-n)} + \frac{(1-\gamma)(1-1/\gamma)}{\lambda} & 1 + \frac{1-\gamma}{\lambda(\lambda-n)} \end{pmatrix}$$

and

$$c = \frac{(\gamma-1)^2(\delta-1)^2}{\gamma\delta}.$$

Again, a polynomial of degree n is uniquely determined by $n+1$ points, and we have calculated $p_{\mathbf{R}}(\lambda)$ in all but two points, which completes the proof. The case $n = 4$ is analogous, and

$$p_{\mathbf{Q}}(\lambda) = \lambda^4 - 4\lambda^3 - 2\left(\gamma + \delta + \frac{1}{\gamma} + \frac{1}{\delta} - 4\right)\lambda - \frac{(\gamma-1)^2(\delta-1)^2}{\gamma\delta}$$

is resulted in. □

Proposition 3. *The principle right eigenvector of a double perturbed PCM can be written in explicit ways.*

In Case 1 (γ and δ are in the same row), the formulas for the principal right eigenvector are the following:

$$\mathbf{w}^{EM} = \begin{pmatrix} \delta\gamma\lambda(\lambda-n+1) \\ \frac{1}{x_1}\left[\gamma\lambda - (n-2)\gamma + \delta + (n-3)\delta\gamma\right] \\ \frac{1}{x_2}\left[\delta\lambda - (n-2)\delta + \gamma + (n-3)\delta\gamma\right] \\ \frac{1}{x_3}\left[\gamma + \delta + \delta\gamma\lambda - 2\delta\gamma\right] \\ \vdots \\ \frac{1}{x_{i-1}}\left[\gamma + \delta + \delta\gamma\lambda - 2\delta\gamma\right] \\ \vdots \\ \frac{1}{x_{n-1}}\left[\gamma + \delta + \delta\gamma\lambda - 2\delta\gamma\right] \end{pmatrix}, \tag{9}$$

$$\mathbf{w}^{EM} = \begin{pmatrix} x_1\gamma\lambda\left[\delta\lambda - (n-2)\delta + \gamma + n - 3\right] \\ \gamma\lambda^3 - (n-1)\gamma\lambda^2 - (n-3)(\gamma^2 - 2\gamma + 1) \\ \frac{x_1}{x_2}\left[\gamma\lambda^2 - \gamma\lambda + \delta\lambda + (n-3)(\delta\gamma - \delta - \gamma + 1)\right] \\ \frac{x_1}{x_3}\left[\gamma\lambda^2 - \gamma\lambda - \gamma + \delta + \delta\gamma\lambda - \delta\gamma + \gamma^2\right] \\ \vdots \\ \frac{x_1}{x_{i-1}}\left[\gamma\lambda^2 - \gamma\lambda - \gamma + \delta + \delta\gamma\lambda - \delta\gamma + \gamma^2\right] \\ \vdots \\ \frac{x_1}{x_{n-1}}\left[\gamma\lambda^2 - \gamma\lambda - \gamma + \delta + \delta\gamma\lambda - \delta\gamma + \gamma^2\right] \end{pmatrix}, \tag{10}$$

$$
\mathbf{w}^{EM} = \begin{pmatrix}
x_2\delta\lambda\left[\delta+\gamma\lambda-(n-2)\gamma+n-3\right] \\
\frac{x_2}{x_1}\left[\delta\lambda^2-\delta\lambda+\gamma\lambda+(n-3)(\delta\gamma-\delta-\gamma+1)\right] \\
\delta\lambda^3-(n-1)\delta\lambda^2-(n-3)(\delta^2-2\delta+1) \\
\frac{x_2}{x_3}\left[\delta\lambda^2-\delta\lambda+\gamma-\delta+\delta^2+\delta\gamma\lambda-\delta\gamma\right] \\
\vdots \\
\frac{x_2}{x_{i-1}}\left[\delta\lambda^2-\delta\lambda+\gamma-\delta+\delta^2+\delta\gamma\lambda-\delta\gamma\right] \\
\vdots \\
\frac{x_2}{x_{n-1}}\left[\delta\lambda^2-\delta\lambda+\gamma-\delta+\delta^2+\delta\gamma\lambda-\delta\gamma\right]
\end{pmatrix}, \tag{11}
$$

$$
\mathbf{w}^{EM} = \begin{pmatrix}
x_3\delta\gamma\lambda(\delta+\gamma+\lambda-2) \\
\frac{x_3}{x_1}\left[\delta\gamma\lambda^2-\delta\gamma\lambda+\gamma^2+\gamma\lambda-\gamma-\delta\gamma+\delta\right] \\
\frac{x_3}{x_2}\left[\delta\gamma\lambda^2-\delta\gamma\lambda-\delta\gamma+\gamma+\delta^2+\delta\lambda-\delta\right] \\
\delta\gamma\lambda^2-4\delta\gamma+\gamma+\delta+\delta^2\gamma+\gamma^2\delta \\
\frac{x_3}{x_4}\left[\delta\gamma\lambda^2-4\delta\gamma+\gamma+\delta+\delta^2\gamma+\gamma^2\delta\right] \\
\vdots \\
\frac{x_3}{x_{n-1}}\left[\delta\gamma\lambda^2-4\delta\gamma+\gamma+\delta+\delta^2\gamma+\gamma^2\delta\right]
\end{pmatrix}. \tag{12}
$$

Formulas (9)–(12) give the same principal right eigenvector, up to a scalar multiplier.

In Case 2A (γ and δ are in different rows, and matrix size is 4×4) the formulas take the following form:

$$
\mathbf{w}^{EM} = \begin{pmatrix}
\delta(\lambda^3\gamma-3\lambda^2\gamma-1+2\gamma-\gamma^2) \\
\frac{1}{x_1}\left[\lambda^2\gamma-2\lambda\gamma+\delta+2\lambda\delta\gamma-2\delta\gamma+\delta\gamma^2\right] \\
\frac{1}{x_2}\gamma\left[\gamma+\lambda-1+\delta\lambda^2-2\lambda\delta+\delta+\lambda\delta\gamma-\delta\gamma\right] \\
\frac{1}{x_3}\left[1+\lambda\gamma-\gamma+\lambda\delta-\delta+\delta\gamma\lambda^2-2\lambda\delta\gamma+\delta\gamma\right]
\end{pmatrix}, \tag{13}
$$

$$
\mathbf{w}^{EM} = \begin{pmatrix}
x_1[\delta\gamma\lambda^2-2\lambda\delta\gamma+1+2\lambda\gamma-2\gamma+\gamma^2] \\
\lambda^3\gamma-3\lambda^2\gamma-1+2\gamma-\gamma^2 \\
\frac{x_1}{x_2}\gamma\left[\lambda\gamma+\lambda^2-2\lambda-\gamma+1+\lambda\delta-\delta+\delta\gamma\right] \\
\frac{x_1}{x_3}\left[\lambda+\lambda^2\gamma-2\lambda\gamma-1+\gamma+\delta+\lambda\delta\gamma-\delta\gamma\right]
\end{pmatrix}, \tag{14}
$$

$$
\mathbf{w}^{EM} = \begin{pmatrix}
x_2\delta(1+\lambda\gamma-\gamma)(\delta+\lambda-1) \\
\frac{x_2}{x_1}\left[1+\lambda\gamma-\gamma+\lambda\delta-\delta+\delta\gamma\lambda^2-2\lambda\delta\gamma+\delta\gamma\right] \\
\gamma(\delta\lambda^3-3\delta\lambda^2-1+2\delta-\delta^2) \\
\frac{x_2}{x_3}\left[2\lambda\delta\gamma+\delta\lambda^2-2\lambda\delta-2\delta\gamma+\gamma+\delta^2\gamma\right]
\end{pmatrix}, \tag{15}
$$

$$
\mathbf{w}^{EM} = \begin{pmatrix}
x_3\delta(\lambda\gamma+\lambda^2-2\lambda-\gamma+1+\lambda\delta-\delta+\delta\gamma) \\
\frac{x_3}{x_1}\left[\gamma+\lambda-1+\delta\lambda^2-2\lambda\delta+\delta+\lambda\delta\gamma-\delta\gamma\right] \\
\frac{x_3}{x_2}\left[2\lambda\delta+\delta\gamma\lambda^2-2\lambda\delta\gamma-2\delta+1+\delta^2\right] \\
\delta\lambda^3-3\delta\lambda^2-1+2\delta-\delta^2
\end{pmatrix}. \tag{16}
$$

Again, formulas (13)–(16) give the same principal right eigenvector, up to a scalar multiplier.

In Case 2B (γ and δ are in different rows, and matrix size is at least 5×5) the formulas are the following:

$$\mathbf{w}^{EM} = \begin{pmatrix} \delta\lambda[\lambda^3\gamma - (n-1)\lambda^2\gamma - (n-3)(\gamma^2 - 2\gamma + 1)] \\ \frac{1}{x_1}\left\{\lambda^3\gamma-(n-2)\lambda^2\gamma+(n-2)\delta\gamma\lambda^2+[\lambda\delta+(n-4)(\delta-1)](\gamma^2-2\gamma+1)\right\} \\ \frac{1}{x_2}\gamma\lambda\left[\gamma + \lambda - 1 + \delta\lambda^2 - 2\lambda\delta + \delta + \lambda\delta\gamma - \delta\gamma\right] \\ \frac{1}{x_3}\lambda\left[1 + \lambda\gamma - \gamma + \lambda\delta - \delta + \delta\gamma\lambda^2 - 2\lambda\delta\gamma + \delta\gamma\right] \\ \frac{1}{x_4}\left[\gamma^2-2\gamma+\lambda^2\gamma+1+\lambda\delta-\delta\gamma\lambda^2-2\lambda\delta\gamma+\lambda\gamma^2\delta+\lambda^3\delta\gamma-\delta+2\delta\gamma-\delta\gamma^2\right] \\ \vdots \\ \frac{1}{x_{n-1}}\left[\gamma^2-2\gamma+\lambda^2\gamma+1+\lambda\delta-\delta\gamma\lambda^2-2\lambda\delta\gamma+\lambda\gamma^2\delta+\lambda^3\delta\gamma-\delta+2\delta\gamma-\delta\gamma^2\right] \end{pmatrix}, \tag{17}$$

$$\mathbf{w}^{EM} = \begin{pmatrix} x_1[\lambda^3\delta\gamma-(n-2)\delta\gamma\lambda^2-(n-4)\delta(\gamma-1)^2+\lambda+(n-2)\lambda^2\gamma-2\lambda\gamma+\lambda\gamma^2+(n-4)(\gamma-1)^2] \\ \lambda(\lambda^3\gamma - (n-1)\lambda^2\gamma - (n-3)(\gamma-1)^2) \\ \frac{x_1}{x_2}\gamma\lambda(\lambda\gamma + \lambda^2 - 2\lambda - \gamma + 1 + \delta\lambda - \delta + \delta\gamma) \\ \frac{x_1}{x_3}\lambda(\lambda + \lambda^2\gamma - 2\lambda\gamma - 1 + \gamma + \delta + \lambda\delta\gamma - \delta\gamma) \\ \frac{x_1}{x_4}(\lambda\gamma^2-2\lambda\gamma+\lambda^3\gamma+\lambda-\gamma^2+2\gamma-\lambda^2\gamma-1+\delta-2\delta\gamma+\delta\gamma^2+\delta\gamma\lambda^2) \\ \vdots \\ \frac{x_1}{x_{n-1}}(\lambda\gamma^2-2\lambda\gamma+\lambda^3\gamma+\lambda-\gamma^2+2\gamma-\lambda^2\gamma-1+\delta-2\delta\gamma+\delta\gamma^2+\delta\gamma\lambda^2) \end{pmatrix}, \tag{18}$$

$$\mathbf{w}^{EM} = \begin{pmatrix} x_2\delta\lambda(1 + \lambda\gamma - \gamma)(\delta + \lambda - 1) \\ \frac{x_2}{x_1}\lambda(1 + \lambda\gamma - \gamma)(1 + \delta\lambda - \delta) \\ \gamma\lambda\left[\lambda^3\delta - (n-1)\delta\lambda^2 - (n-3)(\delta-1)^2\right] \\ \frac{x_2}{x_3}[\delta\lambda^3-(n-2)\delta\lambda^2(1-\gamma)-2\lambda\delta\gamma+2(n-4)\delta(1-\gamma)+\lambda\gamma+\delta^2\lambda\gamma+(n-4)(-1+\gamma-\delta^2+\delta^2\gamma)] \\ \frac{x_2}{x_4}(1 + \lambda\gamma - \gamma)(\delta\lambda^2 + 1 - 2\delta + \delta^2) \\ \vdots \\ \frac{x_2}{x_{n-1}}(1 + \lambda\gamma - \gamma)(\delta\lambda^2 + 1 - 2\delta + \delta^2) \end{pmatrix}, \tag{19}$$

$$\mathbf{w}^{EM} = \begin{pmatrix} x_3\delta\lambda(\lambda\gamma + \lambda^2 - 2\lambda - \gamma + 1 + \delta\lambda - \delta + \delta\gamma) \\ \frac{x_3}{x_1}\lambda(\gamma + \lambda - 1)(1 + \delta\lambda - \delta) \\ \frac{x_3}{x_2}[\lambda^3\delta\gamma-(n-2)\delta\lambda^2(\gamma-1)-2\delta\lambda+2(n-4)\delta(\gamma-1)+\lambda+\delta^2\lambda+(n-4)(1-\gamma+\delta^2-\delta^2\gamma)] \\ \lambda[\delta\lambda^3 - (n-1)\delta\lambda^2 - (n-3)(\delta-1)^2] \\ \frac{x_3}{x_4}(\delta\gamma\lambda^2+\lambda^3\delta-\delta\lambda^2-2\delta\lambda-2\delta\gamma+2\delta-1+\gamma+\lambda+\delta^2\lambda-\delta^2+\delta^2\gamma) \\ \vdots \\ \frac{x_3}{x_{n-1}}(\delta\gamma\lambda^2+\lambda^3\delta-\delta\lambda^2-2\delta\lambda-2\delta\gamma+2\delta-1+\gamma+\lambda+\delta^2\lambda-\delta^2+\delta^2\gamma) \end{pmatrix}, \tag{20}$$

$$\mathbf{w}^{EM} = \begin{pmatrix} x_4\delta\lambda(\gamma^2 - 2\gamma + \lambda^2\gamma + 1)(\delta + \lambda - 1) \\ \frac{x_4}{x_1}\lambda(\gamma^2 - 2\gamma + \lambda^2\gamma + 1)(1 + \delta\lambda - \delta) \\ \frac{x_4}{x_2}\gamma\lambda(\delta\gamma\lambda^2+\lambda^3\delta-\delta\lambda^2-2\delta\lambda-2\delta\gamma+2\delta-1+\gamma+\lambda+\delta^2\lambda-\delta^2+\delta^2\gamma) \\ \frac{x_4}{x_3}\lambda(\delta\lambda^2+\lambda^3\delta\gamma-\delta\gamma\lambda^2-2\lambda\delta\gamma-2\delta+2\delta\gamma-\gamma+1+\lambda\gamma+\delta^2+\delta^2\lambda\gamma-\delta^2\gamma) \\ (\gamma^2 - 2\gamma + \lambda^2\gamma + 1)(\delta\lambda^2 + 1 - 2\delta + \delta^2) \\ \frac{x_4}{x_5}(\gamma^2 - 2\gamma + \lambda^2\gamma + 1)(\delta\lambda^2 + 1 - 2\delta + \delta^2) \\ \vdots \\ \frac{x_4}{x_{n-1}}(\gamma^2 - 2\gamma + \lambda^2\gamma + 1)(\delta\lambda^2 + 1 - 2\delta + \delta^2) \end{pmatrix}. \tag{21}$$

Again, formulas (17)–(21) give the same principal right eigenvector, up to a scalar multiplier.

Proof. The proof is similar to that of the eigenvector formulas (24)–(26) in Farkas (2007). Let us consider Case 1. Let $\mathbf{D} = \mathrm{diag}(1, 1/x_1, \ldots, 1/x_{n-1})$, and let $\mathbf{K_P}(\lambda) =$

$\lambda\mathbf{I} + \mathbf{U}_1\mathbf{V}_1^T - \mathbf{e}\mathbf{e}^T$, with $\mathbf{U}_1$ and $\mathbf{V}_1$ as defined by (7). Since $\mathbf{D}$ is invertible, every column of the one rank matrix $\mathbf{D}\,\mathrm{adj}(\mathbf{K_P}(\lambda_{\max}))\mathbf{D}^{-1}$ is a Perron eigenvector of $\mathbf{P}$.

For Case 2, replace $\mathbf{U}_1$ by $\mathbf{U}_2$ and $\mathbf{V}_1$ by $\mathbf{V}_2$ as defined by (8). □

Remark 5. *Formulas (9)–(21) are positive.*

Proof. It is sufficient to prove the positivity of any arbitrary element of each formula, because the Perron–Frobenius theorem then guarantees the positivity for the vectors as well. The conclusions of the proofs generally follow from $x_i > 0$ for all $i = 1, \ldots, n$, $\gamma, \delta > 0$ and $\lambda > n \geq 4$ (or $n \geq 5$ in Case 2B). The proof for each formula follows:

Formula (9): Positivity is apparent for w_1^{EM}.

Formula (10):

$$\begin{aligned} w_1^{EM} &= x_1\gamma\lambda\left[\delta\lambda - (n-2)\delta + \gamma + n - 3\right] \\ &= x_1\gamma\lambda\left[\delta(\lambda - n + 2) + \gamma + (n-3)\right]. \end{aligned}$$

Formula (11):

$$\begin{aligned} w_1^{EM} &= x_2\delta\lambda\left[\delta + \gamma\lambda - (n-2)\gamma + n - 3\right] \\ &= x_2\delta\lambda\left[\delta + \gamma(\lambda - n + 2) + (n-3)\right]. \end{aligned}$$

Formula (12): Positivity is apparent for w_1^{EM}.

Formula (13):

$$\begin{aligned} w_2^{EM} &= \frac{1}{x_1}\left[\lambda^2\gamma - 2\lambda\gamma + \delta + 2\lambda\delta\gamma - 2\delta\gamma + \delta\gamma^2\right] \\ &= \frac{1}{x_1}\left[\lambda\gamma(\lambda-2) + \delta + 2\delta\gamma(\lambda-1) + \delta\gamma^2\right]. \end{aligned}$$

Formula (14):

$$\begin{aligned} w_1^{EM} &= x_1[\delta\gamma\lambda^2 - 2\lambda\delta\gamma + 1 + 2\lambda\gamma - 2\gamma + \gamma^2] \\ &= x_1[\delta\gamma\lambda(\lambda-2) + 1 + 2\gamma(\lambda-1) + \gamma^2]. \end{aligned}$$

Formula (15):

$$\begin{aligned} w_1^{EM} &= x_2\delta(1 + \lambda\gamma - \gamma)(\delta + \lambda - 1) \\ &= x_2\delta[1 + \gamma(\lambda-1)][\delta + (\lambda-1)]. \end{aligned}$$

Formula (16):

$$\begin{aligned} w_3^{EM} &= \frac{x_3}{x_2}\left[2\lambda\delta + \delta\gamma\lambda^2 - 2\lambda\delta\gamma - 2\delta + 1 + \delta^2\right] \\ &= \frac{x_3}{x_2}\left[2\delta(\lambda-1) + \delta\gamma\lambda(\lambda-2) + 1 + \delta^2\right]. \end{aligned}$$

From here on in the proof, $n \geq 5$.

Formula (17): w_3^{EM} in formula (17) is the same as λw_3^{EM} in formula (13), which is already proven to be positive.

Formula (18): w_3^{EM} in formula (18) is the same as λw_3^{EM} in formula (14), which is already proven to be positive.

Formula (19):

$$\begin{aligned}w_1^{EM} &= x_2\delta\lambda(1+\lambda\gamma-\gamma)(\delta+\lambda-1)\\ &= x_2\delta\lambda[1+\gamma(\lambda-1)][\delta+(\lambda-1)].\end{aligned}$$

Formula (20):

$$\begin{aligned}w_2^{EM} &= \frac{x_3}{x_1}\lambda(\gamma+\lambda-1)(1+\delta\lambda-\delta)\\ &= \frac{x_3}{x_1}\lambda[\gamma+(\lambda-1)][1+\delta(\lambda-1)].\end{aligned}$$

Formula (21):

$$\begin{aligned}w_1^{EM} &= x_4\delta\lambda(\gamma^2-2\gamma+\lambda^2\gamma+1)(\delta+\lambda-1)\\ &= x_4\delta\lambda[\gamma^2+\gamma(\lambda^2-2)+1][\delta+(\lambda-1)].\end{aligned}$$

□

Using these formulas, the paper's main result can be obtained through a series of lemmas. Each of these lemmas corresponds to a directed edge in a digraph. Using these results, the direction of certain arcs can be determined. Thus, it will be shown that directed graphs of Cases 1, 2A and 2B are strongly connected. By Theorem 1, efficiency of the principal right eigenvector is implied.

It follows from the positivity of $\mathbf{w}^{EM}$ (see Remark 5), that both sides of the starting inequalities of each lemma can be multiplied by the respective w_i^{EM} without further discussion. Since there are 28 lemmas, the proofs are in the Appendix.

Cases of $\delta = 1$ and $\gamma = 1$ are not covered by Lemmas 1a–3h due to Remark 3.

The first group of lemmas correspond to Case 1 (γ and δ are in the same row), i.e., the double perturbed PCM is written in form (3).

Lemma 1a (Case 1)**.** *$\delta > 1$ and $\delta \geq \gamma \Rightarrow w_1^{EM}/w_2^{EM} < \delta x_1$.*

Proof. Using formula (10),

$$\frac{w_1^{EM}}{w_2^{EM}} = x_1\frac{\gamma\lambda\left(\delta\lambda-(n-2)\delta+\gamma+n-3\right)}{\gamma\lambda^3-(n-1)\gamma\lambda^2-(n-3)(\gamma^2-2\gamma+1)}.$$

Substitute $\lambda = \lambda_{\max}$ in the characteristic polynomial $p_{\mathbf{P}}(\lambda)$ by Proposition 1:

$$(-1)^n\lambda^{n-3}\left(\lambda^3-n\lambda^2-\left(\frac{\gamma}{\delta}+\frac{\delta}{\gamma}\right)-(n-3)\left(\gamma+\delta+\frac{1}{\gamma}+\frac{1}{\delta}\right)+4n-10\right)=0,$$

which can be transformed to

$$\gamma\delta\lambda^3-\gamma\delta n\lambda^2=\gamma^2+\delta^2+(n-3)\left(\gamma^2\delta+\gamma\delta^2+\delta+\gamma\right)-\gamma\delta(4n-10). \tag{22}$$

The statement to be proven is equivalent to

$$\gamma\lambda\left(\delta\lambda+\gamma-(n-2)\delta+n-3\right)<\delta\left(\gamma\lambda^3-(n-1)\gamma\lambda^2-(n-3)(\gamma-1)^2\right).$$

Using (22) this is further equivalent to

$$(\gamma\delta\lambda^3 - \gamma\delta n\lambda^2) + \lambda\left(\gamma\delta(n-2) - \gamma^2 - \gamma n + 3\gamma\right) - \delta(n-3)(\gamma-1)^2 > 0.$$

Now apply further equivalent transformations:

$$\begin{aligned}
&\gamma\lambda\left(\delta(n-2) - \gamma - n + 3\right) + \gamma^2 + \delta^2 - (n-3)(\delta\gamma^2 - 2\delta\gamma + \delta) \\
&\qquad\qquad + (n-3)\left(\delta^2\gamma + \delta\gamma^2 + \delta + \gamma\right) - \delta\gamma(4n-10) > 0 \\
&\gamma\lambda\left((\delta-1)(n-3) + \delta - \gamma\right) + \gamma^2 + \delta^2 \\
&\qquad\qquad + (n-3)\left(\delta^2\gamma + 2\delta\gamma + \gamma\right) - 4\delta\gamma(n-3) - 2\delta\gamma > 0 \\
&\gamma\lambda\left((\delta-1)(n-3) + (\delta-\gamma)\right) + \gamma(n-3)(\delta-1)^2 + (\delta-\gamma)^2 > 0.
\end{aligned}$$

□

Lemma 1b (Case 1)**.** *$\delta < 1$ and $\delta \le \gamma \Rightarrow w_1^{EM}/w_2^{EM} > \delta x_1$.*

Proof. According to formula (10)

$$\frac{w_1^{EM}}{w_2^{EM}} = x_1 \frac{\gamma\lambda\left(\delta\lambda - (n-2)\delta + \gamma + n - 3\right)}{\gamma\lambda^3 - (n-1)\gamma\lambda^2 - (n-3)(\gamma^2 - 2\gamma + 1)}.$$

Transforming (22) similar to Lemma 1a,

$$\gamma\lambda\left((\delta-1)(n-3) + \delta - \gamma\right) + \gamma(n-3)(\delta-1)^2 + (\delta-\gamma)^2 < 0.$$

Transforming this further yields

$$\begin{aligned}
\gamma(\delta-1)(n-3)\left(\lambda + (\delta-1)\right) + \gamma\lambda(\delta-\gamma) + (\delta-\gamma)^2 &< 0 \\
\gamma(\delta-1)(n-3)\left(\lambda + (\delta-1)\right) + (\delta-\gamma)(\gamma(\lambda-1) + \delta) &< 0.
\end{aligned}$$

□

Lemma 1c (Case 1)**.** *$\gamma > 1$ and $\gamma \ge \delta \Rightarrow w_1^{EM}/w_3^{EM} < \gamma x_2$.*

Proof. The proof follows from switching the role of δ and γ in the proof of Lemma 1a. □

Lemma 1d (Case 1)**.** *$\gamma < 1$ and $\gamma \le \delta \Rightarrow w_1^{EM}/w_3^{EM} > \gamma x_2$.*

Proof. The proof follows from switching the role of δ and γ in the proof of Lemma 1b. □

Lemma 1e (Case 1)**.** *$\gamma, \delta > 1 \Rightarrow w_1^{EM}/w_i^{EM} > x_{i-1}$, $i = 4, \ldots, n$.*

Proof. According to formula (9)

$$\frac{w_1^{EM}}{w_i^{EM}} = x_{i-1} \frac{\gamma\delta\lambda(\lambda - n + 1)}{\gamma + \delta + \gamma\delta\lambda - 2\gamma\delta},$$

which means the statement to be proven is equivalent to

$$\gamma\delta\lambda(\lambda - n + 1) > \gamma + \delta + \gamma\delta\lambda - 2\gamma\delta.$$

Further equivalent transformations yield

$$\begin{aligned}
(\gamma\delta\lambda(\lambda - n)) + (2\gamma\delta - \gamma - \delta) &> 0 \\
\gamma\delta\lambda(\lambda - n) + (\delta-1)(\gamma-1) + (\delta\gamma - 1) &> 0.
\end{aligned}$$

□

Lemma 1f (Case 1)**.** $\gamma, \delta < 1 \Rightarrow w_1^{EM}/w_i^{EM} < x_{i-1}$, $i = 4, \ldots, n$.

Proof. According to formula (12)

$$\frac{w_1^{EM}}{w_i^{EM}} = x_{i-1}\frac{\gamma\delta\lambda(\delta+\gamma+\lambda-2)}{\gamma\delta\lambda^2 - 4\gamma\delta + \gamma + \delta + \delta^2\gamma + \gamma^2\delta}.$$

Applying further equivalent transformations

$$\frac{\gamma\delta\lambda(\delta+\gamma+\lambda-2)}{\gamma\delta\lambda^2 - 4\gamma\delta + \gamma + \delta + \delta^2\gamma + \gamma^2\delta} < 1$$

$$\begin{aligned}
\gamma\delta\lambda(\delta+\gamma+\lambda-2) &< \gamma\delta(\lambda^2-4) + \gamma + \delta + \delta^2\gamma + \gamma^2\delta \\
\lambda(\delta+\gamma+\lambda-2) &< \lambda^2 - 4 + \frac{1}{\delta} + \frac{1}{\gamma} + \delta + \gamma \\
0 &< \lambda^2 - 4 + \frac{1}{\delta} + \frac{1}{\gamma} + \delta + \gamma - \lambda\delta - \lambda\gamma - \lambda^2 + 2\lambda \\
0 &< 2(\lambda-2) + (1-\lambda)(\delta+\gamma) + \frac{1}{\delta} + \frac{1}{\gamma} \\
0 &< 2(\lambda-1) - 2 + (1-\lambda)(\delta+\gamma) + \frac{1}{\delta} + \frac{1}{\gamma} \\
0 &< (\lambda-1)(2-\delta-\gamma) + \frac{1}{\delta} + \frac{1}{\gamma} - 2.
\end{aligned}$$

□

Lemma 1g (Case 1)**.** $\delta \lesseqqgtr \gamma \Leftrightarrow w_2^{EM}/w_3^{EM} \gtreqqless x_2/x_1$.

Proof. According to formula (9), we need to consider

$$\frac{w_2^{EM}}{w_3^{EM}} = \frac{x_2}{x_1} \cdot \frac{\gamma\lambda - (n-2)\gamma + \delta + (n-3)\gamma\delta}{\delta\lambda - (n-2)\delta + \gamma + (n-3)\gamma\delta} \gtreqqless 1.$$

Applying further equivalent transformations

$$\begin{aligned}
\gamma\lambda - (n-2)\gamma + \delta + (n-3)\gamma\delta &\gtreqqless \delta\lambda - (n-2)\delta + \gamma + (n-3)\gamma\delta \\
\lambda(\gamma-\delta) - (n-2)(\gamma-\delta) + \delta - \gamma &\gtreqqless 0 \\
(\gamma-\delta)(\lambda-n+1) &\gtreqqless 0.
\end{aligned}$$

The third factor is positive because $\lambda > n$. □

Lemma 1h (Case 1)**.** $\delta \gtrless 1 \Leftrightarrow w_2^{EM}/w_i^{EM} \lessgtr x_{i-1}/x_1$, $i = 4, \ldots, n$.

Proof. According to formula (9)

$$\frac{w_2^{EM}}{w_i^{EM}} = \frac{x_{i-1}}{x_1} \cdot \frac{\gamma\lambda - (n-2)\gamma + \delta + (n-3)\gamma\delta}{\gamma + \delta + \gamma\delta\lambda - 2\gamma\delta}.$$

Equivalent transformations yield

$$\begin{aligned}
\gamma\lambda - (n-2)\gamma + \delta + (n-3)\gamma\delta &< \gamma + \delta + \gamma\delta\lambda - 2\gamma\delta \\
0 &< \gamma(\delta-1)(\lambda-n+1).
\end{aligned}$$

The third factor is positive because $\lambda > n$. □

Lemma 1i (Case 1)**.** $\gamma \gtreqless 1 \Leftrightarrow w_3^{EM}/w_i^{EM} \lesseqgtr x_{i-1}/x_2$, $i = 4, \ldots, n$.

Proof. The proof follows from switching the role of δ and γ in the proof of Lemma 1h. □

Lemma 1j (Case 1)**.** $w_i^{EM}/w_j^{EM} = x_{j-1}/x_{i-1}$, $i, j = 4, \ldots, n$.

Proof. It follows from each of formulas (9)–(12). □

Corollary 1. *There exists a directed cycle in each graph corresponding to Case 1 (Figure 2):*

$$\begin{aligned}
&\delta > 1, \gamma > \delta : 1 \to i \to 2 \to 3 \to 1,\\
&\gamma > 1, \gamma < \delta : 1 \to i \to 3 \to 2 \to 1,\\
&\delta > 1, \gamma < 1 : 1 \to 3 \to i \to 2 \to 1,\\
&\delta < 1, \gamma < \delta : 1 \to 3 \to 2 \to i \to 1,\\
&\gamma < 1, \gamma > \delta : 1 \to 2 \to 3 \to i \to 1,\\
&\delta < 1, \gamma > 1 : 1 \to 2 \to i \to 3 \to 1.
\end{aligned}$$

The second group of lemmas correspond to Case 2A (γ and δ are in different rows, and matrix size is 4×4), i.e., the double perturbed PCM is written in form (4).

Lemma 2a (Case 2A)**.** $\delta \gtreqless 1 \Leftrightarrow w_1^{EM}/w_2^{EM} \lesseqgtr \delta x_1$.

Proof. Formula (16) is used for this proof. Multiplying both sides by w_2^{EM}, the statement to be proven can be written as:

$$\begin{aligned}
&x_3\delta(\lambda\gamma + \lambda^2 - 2\lambda - \gamma + 1 + \lambda\delta - \delta + \delta\gamma)\\
&\qquad \lesseqgtr \delta x_1 \frac{x_3}{x_1}\left(\gamma + \lambda - 1 + \delta\lambda^2 - 2\lambda\delta + \delta + \lambda\delta\gamma - \delta\gamma\right). \quad (23)
\end{aligned}$$

Further equivalent transformations yield:

$$\begin{aligned}
0 &\lesseqgtr \lambda^2\delta - \lambda^2 + 3\lambda - \lambda\gamma - 3\lambda\delta + \lambda\delta\gamma - 2\delta\gamma + 2\gamma + 2\delta - 2\\
0 &\lesseqgtr \lambda^2(\delta - 1) + \lambda\gamma(\delta - 1) + 3\lambda(1 - \delta) + 2\gamma(1 - \delta) + 2(\delta - 1)\\
0 &\lesseqgtr (\delta - 1)(\lambda(\lambda - 3) + \gamma(\lambda - 2) + 2).
\end{aligned}$$

The second factor on the right hand side is always positive because $\lambda > n = 4$ and $\gamma, \delta > 0$. □

Lemma 2b (Case 2A)**.** $\delta > 1, \gamma < 1 \Rightarrow w_1^{EM}/w_3^{EM} > x_2$.

Proof. Formula (14) is used in this proof. Multiplying both sides by w_3^{EM}, the statement of the lemma is equivalent to:

$$\begin{aligned}
&x_1(\delta\gamma\lambda^2 - 2\lambda\delta\gamma + 1 + 2\lambda\gamma - 2\gamma + \gamma^2)\\
&\qquad < x_2\frac{x_1}{x_2}\gamma\left(\lambda\gamma + \lambda^2 - 2\lambda - \gamma + 1 + \lambda\delta - \delta + \delta\gamma\right).
\end{aligned}$$

Further equivalent transformations yield:

$$0 < \lambda^2\gamma - \lambda^2\gamma\delta - 4\lambda\gamma + \lambda\gamma^2 + 3\lambda\delta\gamma - 2\gamma^2 + 3\gamma + \delta\gamma^2 - \delta\gamma - 1$$

$$
\begin{aligned}
&0 < \lambda^2\gamma(1-\delta) + \lambda\gamma(\gamma-1) + 3\lambda\gamma(\delta-1) + (\gamma-1) + 2\gamma(1-\gamma) + \delta\gamma(\gamma-1) \\
&0 < (1-\delta)\lambda\gamma(\lambda-3) + (\gamma-1)(\gamma(\lambda-2) + \delta\gamma + 1). \qquad (24)
\end{aligned}
$$

The second factor on the right hand side is always positive because $\lambda > n = 4$ and $\gamma, \delta > 0$. ☐

Lemma 2c (Case 2A)**.** $\delta < 1, \gamma > 1 \Rightarrow w_1^{EM}/w_3^{EM} < x_2$.

Proof. The proof follows from the right hand side of (24) being positive in the case of $\delta < 1, \gamma > 1$. ☐

Lemma 2d (Case 2A)**.** $\delta, \gamma < 1 \Rightarrow w_1^{EM}/w_4^{EM} < x_3$.

Proof. Again, formula (14) is used for this proof. Multiplying both sides by w_4^{EM}, the statement to be proven is equivalent to:

$$
\begin{aligned}
&x_1(\delta\gamma\lambda^2 - 2\lambda\delta\gamma + 1 + 2\lambda\gamma - 2\gamma + \gamma^2) \\
&\qquad < x_3\frac{x_1}{x_3}\left(\lambda + \lambda^2\gamma - 2\lambda\gamma - 1 + \gamma + \delta + \lambda\delta\gamma - \delta\gamma\right).
\end{aligned}
$$

Further equivalent transformations yield:

$$
\begin{aligned}
\lambda^2\gamma\delta - \lambda^2\gamma + 4\lambda\gamma - 3\lambda\delta\gamma - \lambda + \gamma^2 - 3\gamma + \delta\gamma - \delta + 2 &< 0 \\
(\delta-1)(\lambda^2\gamma - 3\lambda\gamma) + (\gamma-1)(\lambda + \gamma - 2 + \delta) &< 0 \\
(\delta-1)\lambda\gamma(\lambda-3) + (\gamma-1)((\lambda-2) + \gamma + \delta) &< 0. \qquad (25)
\end{aligned}
$$

The left hand side is negative if $\gamma, \delta < 1$, because $\lambda > n = 4$. ☐

Lemma 2e (Case 2A)**.** $\delta, \gamma > 1 \Rightarrow w_1^{EM}/w_4^{EM} > x_3$.

Proof. The proof follows from the left hand side of (25) being positive if $\gamma, \delta > 1$. ☐

Lemma 2f (Case 2A)**.** $\delta, \gamma < 1 \Rightarrow w_2^{EM}/w_3^{EM} > x_2/x_1$.

Proof. Formula (13) is used in this proof. Multiplying both sides by w_3^{EM}, the statement of the lemma can be written as:

$$
\begin{aligned}
&\frac{1}{x_1}\left(\lambda^2\gamma - 2\lambda\gamma + \delta + 2\lambda\delta\gamma - 2\delta\gamma + \delta\gamma^2\right) \\
&\qquad > \frac{x_2}{x_1}\frac{1}{x_2}\gamma\left(\gamma + \lambda - 1 + \delta\lambda^2 - 2\lambda\delta + \delta + \lambda\delta\gamma - \delta\gamma\right).
\end{aligned}
$$

Further equivalent transformations yield:

$$
\begin{aligned}
&0 > \lambda^2\gamma\delta - \lambda^2\gamma - 4\lambda\delta\gamma + 3\lambda\gamma + \lambda\delta\gamma^2 + \gamma^2 - 2\delta\gamma^2 + 3\delta\gamma - \gamma - \delta \\
&0 > (\delta-1)(\lambda^2\gamma - 3\lambda\gamma) + (\gamma-1)(\lambda\delta\gamma - 2\delta\gamma + \delta + \gamma) \\
&0 > (\delta-1)\lambda\gamma(\lambda-3) + (\gamma-1)(\delta\gamma(\lambda-2) + \delta + \gamma). \qquad (26)
\end{aligned}
$$

The right hand side is negative if $\delta, \gamma < 1$, because $\lambda > n = 4$. ☐

Lemma 2g (Case 2A)**.** $\delta, \gamma > 1 \Rightarrow w_2^{EM}/w_3^{EM} < x_2/x_1$.

Proof. The proof follows from the right hand side of (26) being positive if $\delta, \gamma > 1$. $\square$

Lemma 2h (Case 2A)**.** $\delta < 1, \gamma > 1 \Rightarrow w_2^{EM}/w_4^{EM} > x_3/x_1$.

Proof. Again, formula (13) is used in this proof. Multiplying both sides by w_4^{EM}, the statement to be proven is equivalent to:

$$\frac{1}{x_1}\left(\lambda^2\gamma - 2\lambda\gamma + \delta + 2\lambda\delta\gamma - 2\delta\gamma + \delta\gamma^2\right)$$
$$> \frac{x_3}{x_1}\frac{1}{x_3}\left(1 + \lambda\gamma - \gamma + \lambda\delta - \delta + \delta\gamma\lambda^2 - 2\lambda\delta\gamma + \delta\gamma\right).$$

Further equivalent transformations yield:

$$\begin{aligned}
&0 > \lambda^2\gamma\delta - \lambda^2\gamma - 4\lambda\delta\gamma + 3\lambda\gamma + \lambda\delta + 3\delta\gamma - \delta\gamma^2 - 2\delta - \gamma + 1\\
&0 > (\delta - 1)(\lambda^2\gamma - 3\lambda\gamma) + (1 - \gamma)(\lambda\delta - 2\delta + \delta\gamma + 1)\\
&0 > (\delta - 1)\lambda\gamma(\lambda - 3) + (1 - \gamma)(\delta(\lambda - 2) + \delta\gamma + 1). \qquad (27)
\end{aligned}$$

The right hand side of (27) is negative, if $\delta < 1, \gamma > 1$, because $\lambda > n = 4$. $\square$

Lemma 2i (Case 2A)**.** $\delta > 1, \gamma < 1 \Rightarrow w_2^{EM}/w_4^{EM} < x_3/x_1$.

Proof. The proof follows from the right hand side of (27) being positive if $\delta > 1, \gamma < 1$. $\square$

Lemma 2j (Case 2A)**.** $\gamma \gtreqless 1 \Leftrightarrow w_3^{EM}/w_4^{EM} \lesseqgtr \gamma x_3/x_2$.

Proof. Once again, formula (13) is used for the proof. Multiplying both sides by w_4^{EM}, the first statement (for $\gamma > 1$) becomes equivalent to:

$$\frac{1}{x_2}\gamma\left(\gamma + \lambda - 1 + \delta\lambda^2 - 2\lambda\delta + \delta + \lambda\delta\gamma - \delta\gamma\right)$$
$$\lesseqgtr \gamma\frac{x_3}{x_2}\frac{1}{x_3}\left(1 + \lambda\gamma - \gamma + \lambda\delta - \delta + \delta\gamma\lambda^2 - 2\lambda\delta\gamma + \delta\gamma\right). \qquad (28)$$

Applying further equivalent transformations:

$$\begin{aligned}
&0 \lesseqgtr \lambda^2\delta\gamma - \lambda^2\delta - 3\lambda\delta\gamma + 3\lambda\delta + \lambda\gamma - \lambda + 2\delta\gamma - 2\delta - 2\gamma + 2\\
&0 \lesseqgtr (\gamma - 1)(\lambda^2\delta - 3\lambda\delta + \lambda + 2\delta - 2)\\
&0 \lesseqgtr (\gamma - 1)(\lambda\delta(\lambda - 3) + (\lambda - 2) + 2\delta). \qquad (29)
\end{aligned}$$

The second factor on the right hand side of (29) is positive because $\lambda > n = 4$ and $\gamma, \delta > 0$. $\square$

Corollary 2. *There exists a directed cycle in each graph corresponding to Case 2A (Figure 3):*

$$\begin{aligned}
&\delta > 1, \gamma > 1 : 1 \to 4 \to 3 \to 2 \to 1,\\
&\delta > 1, \gamma < 1 : 1 \to 3 \to 4 \to 2 \to 1,\\
&\delta < 1, \gamma < 1 : 1 \to 2 \to 3 \to 4 \to 1,\\
&\delta < 1, \gamma > 1 : 1 \to 2 \to 4 \to 3 \to 1.
\end{aligned}$$

The last group of lemmas correspond to Case 2B, when γ and δ are in different rows, and matrix size is at least 5×5, i.e., the double perturbed PCM is written in form (5).

Lemma 3a (Case 2B)**.** $\gamma \gtreqless 1 \Leftrightarrow w_3^{EM}/w_4^{EM} \lesseqgtr \gamma x_3/x_2$.

Proof. Using formula (17) the proof is similar to the proof of Lemma 2j, the only difference is in (28) where both sides are multiplied by λ, which immediately cancel each other. □

Lemma 3b (Case 2B)**.** $\delta \gtreqless 1 \Leftrightarrow w_1^{EM}/w_2^{EM} \lesseqgtr \delta x_1$.

Proof. Using formula (21) the proof is similar to the proof of Lemma 2a, the only difference is in (23) where both sides (the formula for w_1^{EM} and w_2^{EM}) are multiplied by λ, which immediately cancel each other. This may not be apparent about w_2^{EM}, but

$$(\gamma + \lambda - 1)(1 + \delta\lambda - \delta) = \gamma + \lambda - 1 + \lambda\gamma\delta + \lambda^2\delta - \lambda\delta - \gamma\delta - \lambda\delta + \delta$$

which, after reduction, gives the same formula. □

Lemma 3c (Case 2B)**.** $\delta \gtreqless 1 \Leftrightarrow w_1^{EM}/w_i^{EM} \gtreqless x_{i-1}$, $i = 5, \ldots, n$.

Proof. Formula (19) is used for this proof.

$$\begin{aligned}
x_2\delta\lambda(1 + \lambda\gamma - \gamma)(\delta + \lambda - 1) &\gtreqless x_{i-1}\frac{x_2}{x_{i-1}}(1 + \lambda\gamma - \gamma)(\delta\lambda^2 + 1 - 2\delta + \delta^2) \\
\lambda\delta^2 + \lambda^2\delta - \lambda\delta &\gtreqless \delta\lambda^2 + 1 - 2\delta + \delta^2 \\
\lambda\delta(\delta - 1) + \delta(1 - \delta) + (\delta - 1) &\gtreqless 0 \\
(\delta - 1)(\delta(\lambda - 1) + 1) &\gtreqless 0.
\end{aligned}$$

The second factor on the left hand side is always positive because $\lambda > n \geq 5$ and $\delta > 0$. □

Lemma 3d (Case 2B)**.** $\delta \gtreqless 1 \Leftrightarrow w_2^{EM}/w_i^{EM} \lesseqgtr x_{i-1}/x_1$, $i = 5, \ldots, n$.

Proof. Again, formula (19) is used in the proof.

$$\begin{aligned}
\frac{x_2}{x_1}\lambda(1 + \lambda\gamma - \gamma)(1 + \delta\lambda - \delta) &\lesseqgtr \frac{x_{i-1}}{x_1}\frac{x_2}{x_{i-1}}(1 + \lambda\gamma - \gamma)(\delta\lambda^2 + 1 - 2\delta + \delta^2) \\
\lambda + \lambda^2\delta - \delta\lambda &\lesseqgtr \delta\lambda^2 + 1 - 2\delta + \delta^2 \\
0 &\lesseqgtr \lambda\delta - \lambda + \delta^2 - 2\delta + 1 \\
0 &\lesseqgtr \lambda(\delta - 1) + (\delta - 1)^2 \\
0 &\lesseqgtr (\delta - 1)((\lambda - 1) + \delta).
\end{aligned}$$

The second factor on the right hand side is always positive because $\lambda > n \geq 5$ and $\delta > 0$. □

Lemma 3e (Case 2B)**.** $\gamma \gtreqless 1 \Leftrightarrow w_3^{EM}/w_i^{EM} \gtreqless x_{i-1}/x_2$, $i = 5, \ldots, n$.

Proof. Formula (18) is used in this proof.

$$
\begin{aligned}
&\frac{x_1}{x_2}\gamma\lambda(\lambda\gamma+\lambda^2-2\lambda-\gamma+1+\delta\lambda-\delta+\delta\gamma)\\
&\qquad\gtrless \frac{x_4}{x_2}\frac{x_1}{x_4}(\lambda\gamma^2-2\lambda\gamma+\lambda^3\gamma+\lambda-\gamma^2+2\gamma-\lambda^2\gamma-1+\delta-2\delta\gamma+\delta\gamma^2+\delta\gamma\lambda^2).
\end{aligned}
$$

Further equivalent transformations yield:

$$
\begin{aligned}
\lambda^2\gamma^2-\lambda^2\gamma-2\lambda\gamma^2+3\lambda\gamma+\lambda\gamma^2\delta-\lambda\delta\gamma-\lambda+\gamma^2-2\gamma+1-\delta+2\delta\gamma-\delta\gamma^2 &\gtrless 0\\
(\gamma-1)(\lambda^2\gamma-2\lambda\gamma+\lambda+\lambda\delta\gamma+(\gamma-1)-\delta(\gamma-1)) &\gtrless 0\\
(\gamma-1)(\lambda\gamma(\lambda-2)+(\lambda-1)+\delta\gamma(\lambda-1)+\gamma+\delta) &\gtrless 0.
\end{aligned}
$$

The second factor on the left hand side is always positive because $\lambda > n \geq 5$ and $\gamma, \delta > 0$. □

Lemma 3f.
$\gamma > 1, \delta < 1 \Rightarrow w_2^{EM}/w_4^{EM} > x_3/x_1$.
$\gamma < 1, \delta > 1 \Rightarrow w_2^{EM}/w_4^{EM} < x_3/x_1$.

Proof. Instead of the statement of the lemma, we will prove the following stronger statement:

$$\gamma \gtreqless \delta \Leftrightarrow w_2^{EM}/w_4^{EM} \gtreqless x_3/x_1.$$

Formula (21) is used in this proof.

$$
\begin{aligned}
&\frac{x_4}{x_1}\lambda(\gamma^2-2\gamma+\lambda^2\gamma+1)(1+\delta\lambda-\delta)\\
&\gtreqless \frac{x_3}{x_1}\frac{x_4}{x_3}\lambda(\delta\lambda^2+\lambda^3\delta\gamma-\delta\gamma\lambda^2-2\lambda\delta\gamma-2\delta+2\delta\gamma-\gamma+1+\lambda\gamma+\delta^2+\delta^2\lambda\gamma-\delta^2\gamma).
\end{aligned}
$$

This is further equivalent to

$$
\begin{aligned}
&\gamma^2+\lambda\gamma^2\delta-\gamma^2\delta-2\gamma-2\lambda\gamma\delta+2\gamma\delta+\lambda^2\gamma+\lambda^3\gamma\delta-\lambda^2\gamma\delta+1+\lambda\delta-\delta\\
&\qquad\gtreqless \lambda^2\delta+\lambda^3\gamma\delta-\lambda^2\gamma\delta-2\lambda\gamma\delta-2\delta+2\gamma\delta-\gamma+1+\lambda\gamma+\delta^2+\lambda\gamma\delta^2-\gamma\delta^2.
\end{aligned}
$$

Further equivalent transformations yield

$$\lambda^2\gamma-\lambda^2\delta+\lambda\delta-\lambda\gamma+\lambda\gamma^2\delta-\lambda\gamma\delta^2+\gamma^2-\delta^2+\gamma\delta^2-\gamma^2\delta+2\delta-2\gamma+\gamma-\delta \gtreqless 0$$

$$\lambda^2(\gamma-\delta)+\lambda(\delta-\gamma)+\lambda\gamma\delta(\gamma-\delta)+(\gamma+\delta)(\gamma-\delta)+\gamma\delta(\delta-\gamma)+2(\delta-\gamma)+(\gamma-\delta) \gtreqless 0$$

$$
\begin{aligned}
(\gamma-\delta)(\lambda^2-\lambda+\lambda\gamma\delta+\gamma+\delta-\gamma\delta-1) &\gtreqless 0\\
(\gamma-\delta)(\lambda^2-2\lambda+\lambda\gamma\delta-\gamma\delta+\lambda-1+\gamma+\delta) &\gtreqless 0\\
(\gamma-\delta)(\lambda(\lambda-2)+\gamma\delta(\lambda-1)+(\lambda-1)+\gamma+\delta) &\gtreqless 0.
\end{aligned}
$$

The second factor on the left hand side is always positive because $\lambda > n \geq 5$ and $\gamma, \delta > 0$. □

Lemma 3g (Case 2B)**.**
$\gamma, \delta > 1 \Rightarrow w_1^{EM}/w_4^{EM} > x_3$.
$\gamma, \delta < 1 \Rightarrow w_1^{EM}/w_4^{EM} < x_3$.

Proof. Instead of the above statement, we will prove the following stronger statement:

$$\gamma\delta \gtreqless 1 \Leftrightarrow w_1^{EM}/w_4^{EM} \gtreqless x_3.$$

Formula (21) is used in this proof.

$$x_4\delta\lambda(\gamma^2 - 2\gamma + \lambda^2\gamma + 1)(\delta + \lambda - 1)$$
$$\gtreqless x_3\frac{x_4}{x_3}\lambda(\delta\lambda^2 + \lambda^3\delta\gamma - \delta\gamma\lambda^2 - 2\lambda\delta\gamma - 2\delta + 2\delta\gamma - \gamma + 1 + \lambda\gamma + \delta^2 + \delta^2\lambda\gamma - \delta^2\gamma).$$

Further equivalent transformations yield:

$$\lambda^2\delta^2\gamma - \lambda^2\delta + \lambda\gamma^2\delta - \lambda\gamma\delta^2 + \lambda\delta - \lambda\gamma + \gamma^2\delta^2 - \gamma\delta^2 - \delta\gamma^2 + \delta + \gamma - 1 \gtreqless 0$$
$$(\delta\gamma - 1)(\lambda^2\delta + \lambda\gamma - \lambda\delta + \delta\gamma + 1 - \delta - \gamma) \gtreqless 0$$
$$(\delta\gamma - 1)(\lambda\delta(\lambda - 2) + \gamma(\lambda - 1) + \delta(\lambda - 1) + \delta\gamma + 1) \gtreqless 0.$$

The second factor is always positive because $\lambda > n \geq 5$ and $\gamma, \delta > 0$. The first factor is positive exactly if $\gamma\delta > 1$, and negative exactly if $\gamma\delta < 1$. □

Lemma 3h (Case 2B)**.** $w_i^{EM}/w_j^{EM} = x_{j-1}/x_{i-1}$, $i, j = 5, \ldots, n$.

Proof. It follows from each of formulas (17)–(21). □

Corollary 3. *There exists a directed cycle in each graph corresponding to Case 2B (Figure 4):*

$$\delta > 1, \gamma > 1 : 1 \to 4 \to 3 \to i \to 2 \to 1,$$
$$\delta > 1, \gamma < 1 : 1 \to i \to 3 \to 4 \to 2 \to 1,$$
$$\delta < 1, \gamma < 1 : 1 \to 2 \to i \to 3 \to 4 \to 1,$$
$$\delta < 1, \gamma > 1 : 1 \to 2 \to 4 \to 3 \to i \to 1.$$