

An inconsistency control system based on incomplete pairwise comparison matrices

Sándor Bozóki

Computer and Automation Research Institute,
Hungarian Academy of Sciences (MTA SZTAKI);
Corvinus University of Budapest (BCE)

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Outline

- Incomplete pairwise comparison matrix
- Eigenvalue optimization
- Cyclic coordinates and Newton's method by using Harker's formulas for derivatives
- CR inconsistency prediction
- Koczkodaj's CM inconsistency index
- Ordinal inconsistency

Incomplete pairwise comparison matrix
 (= pairwise comparison matrix with missing elements)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & * & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & * \\ * & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & * & 1/a_{3n} & \dots & 1 \end{pmatrix} .$$

Incomplete pairwise comparison matrix
 (= pairwise comparison matrix with missing elements)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & x_d \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/x_d & 1/a_{3n} & \dots & 1 \end{pmatrix},$$

where $x_1, x_2, \dots, x_d \in \mathbb{R}_+$.

Based on the idea above, Shiraishi, Obata and Daigo considered the eigenvalue optimization problems as follows.

In case of one missing element, denoted by x , the $\lambda_{max}(\mathbf{A}(x))$ to be minimized:

$$\min_{x>0} \lambda_{max}(\mathbf{A}(x)).$$

In case of more than one missing elements, arranged in vector \mathbf{x} , the aim is to solve

$$\min_{\mathbf{x}>\mathbf{0}} \lambda_{max}(\mathbf{A}(\mathbf{x})).$$

Graph representation of a pairwise comparison matrix

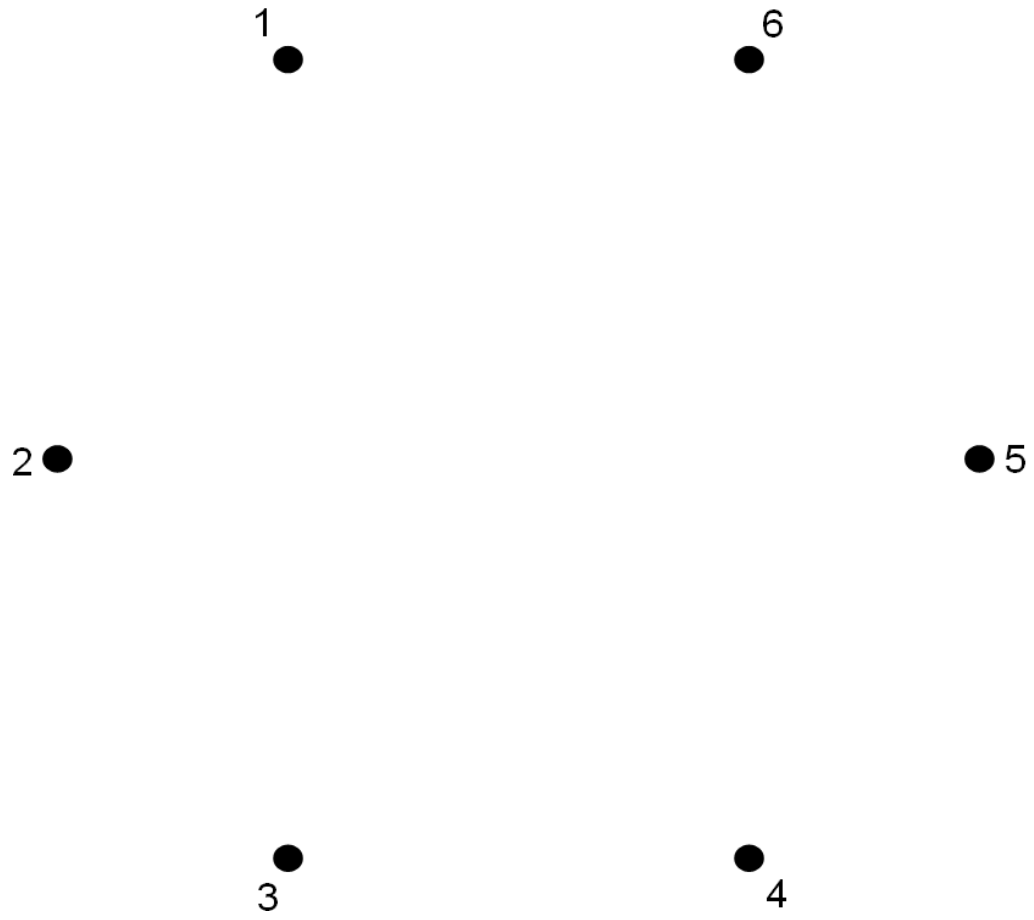
Given A incomplete pairwise comparison matrix of size $n \times n$. Graph $G = (V, E)$ is defined as follows:

$$V = \{1, 2, \dots, n\}$$

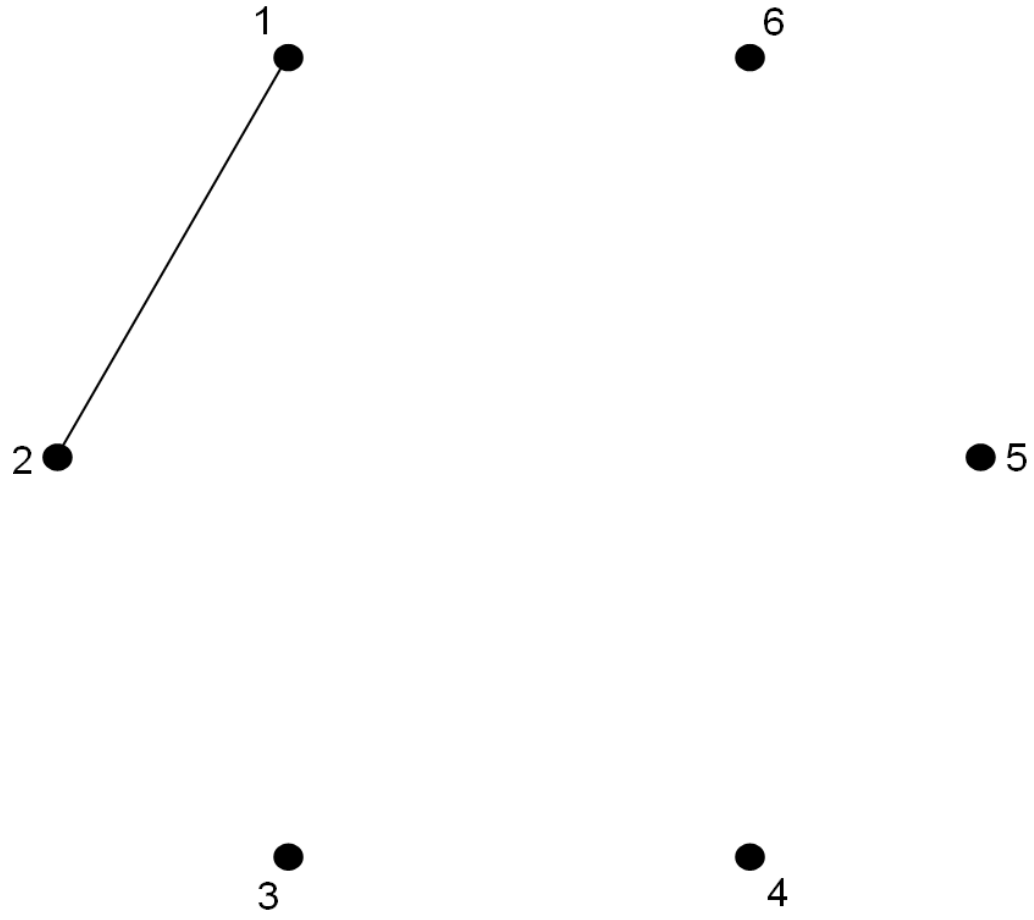
$$E = \{e(i, j) \mid a_{ij} \text{ (and } a_{ji}) \text{ are given and } i \neq j\}$$

Special case: all the comparisons are given, the corresponding graph is K_n .

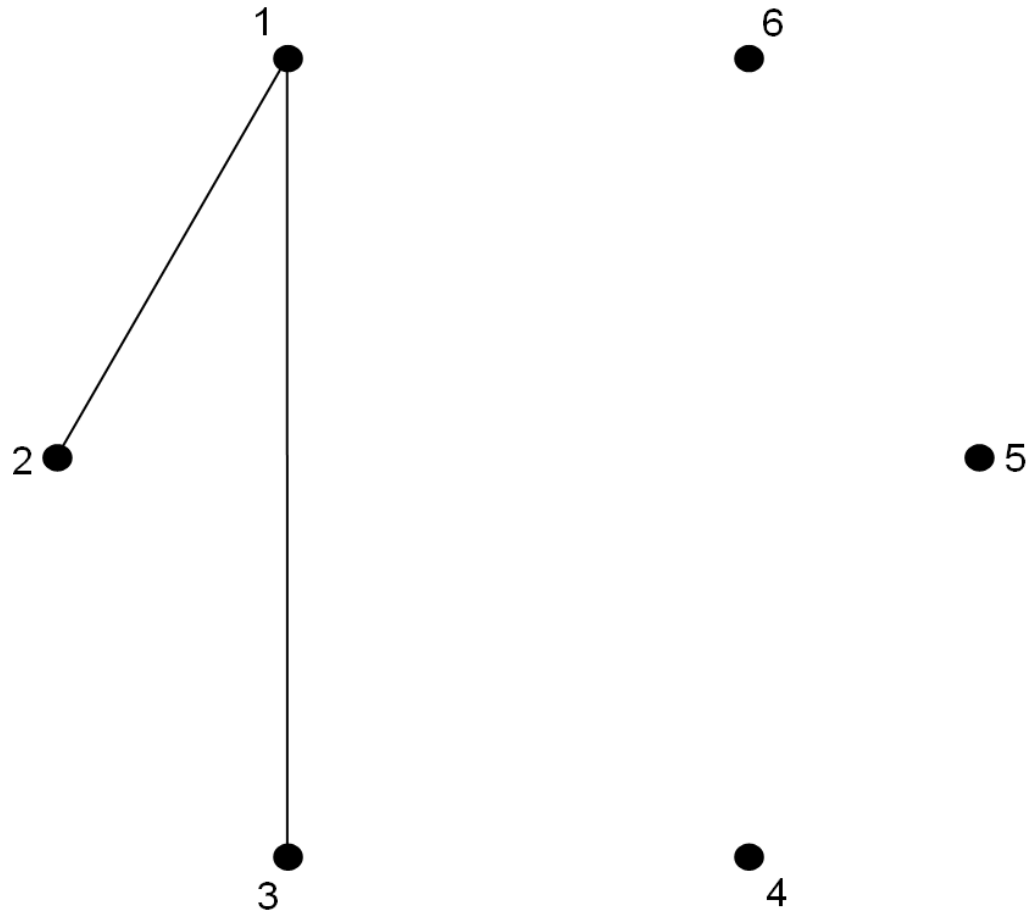
$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$



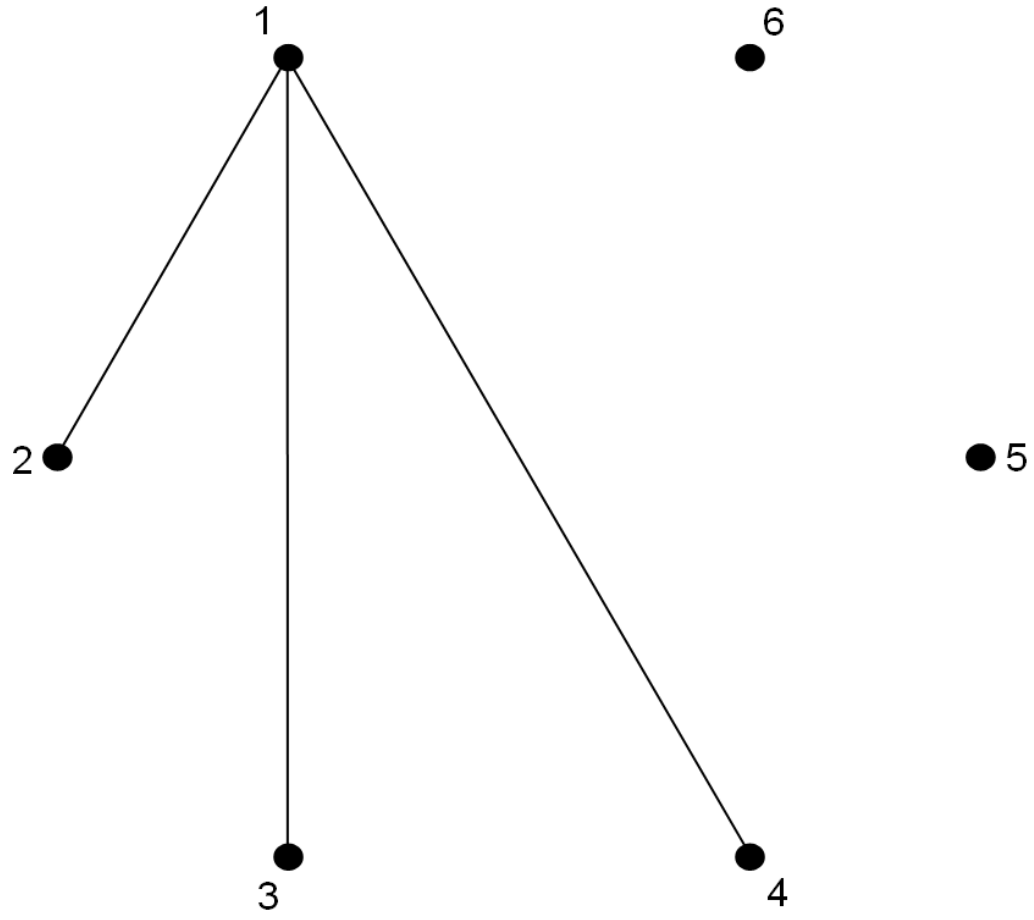
$$\begin{pmatrix} 1 & 2 \\ 1/2 & 1 \\ & 1 \\ & & 1 \\ & & & 1 \\ & & & & 1 \end{pmatrix}$$



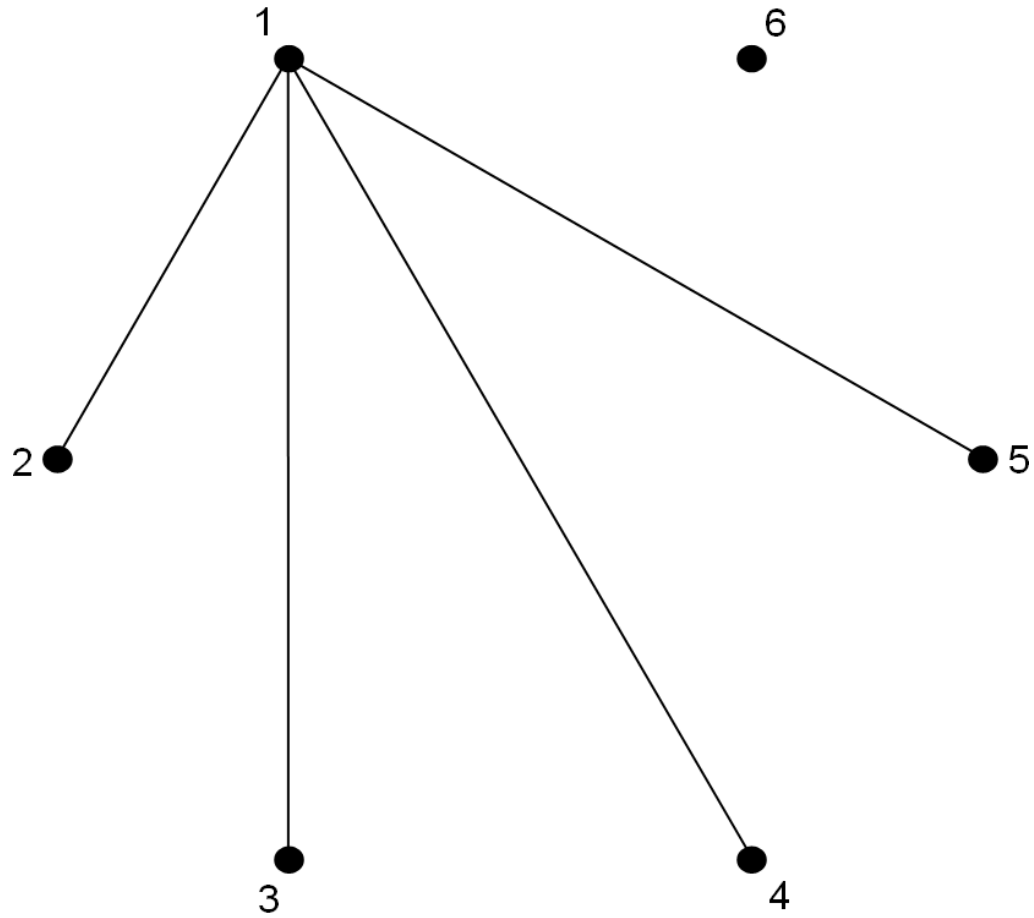
$$\begin{pmatrix} 1 & 2 & 5 \\ 1/2 & 1 & \\ 1/5 & & 1 \\ & 1 & & 1 \\ & & 1 & & 1 \end{pmatrix}$$



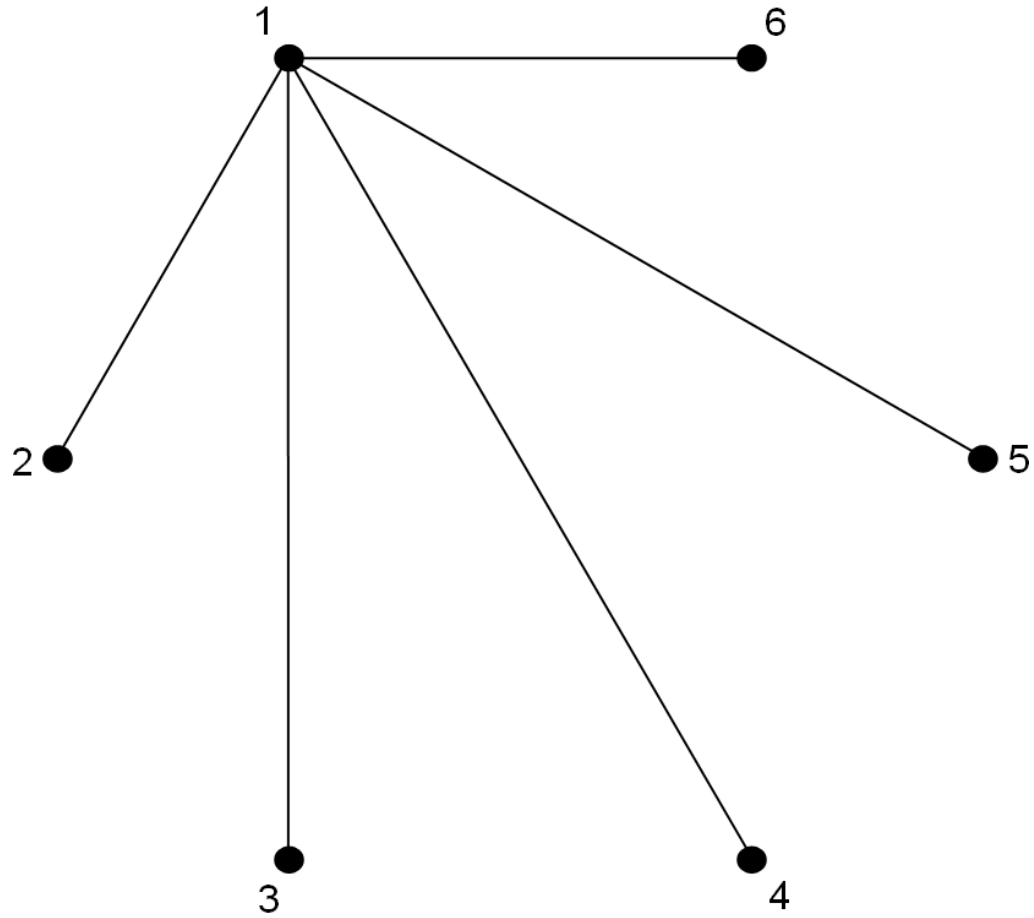
$$\begin{pmatrix} 1 & 2 & 5 & 7 \\ 1/2 & 1 & & \\ 1/5 & & 1 & \\ 1/7 & & & 1 \\ & & & & 1 \\ & & & & & 1 \end{pmatrix}$$



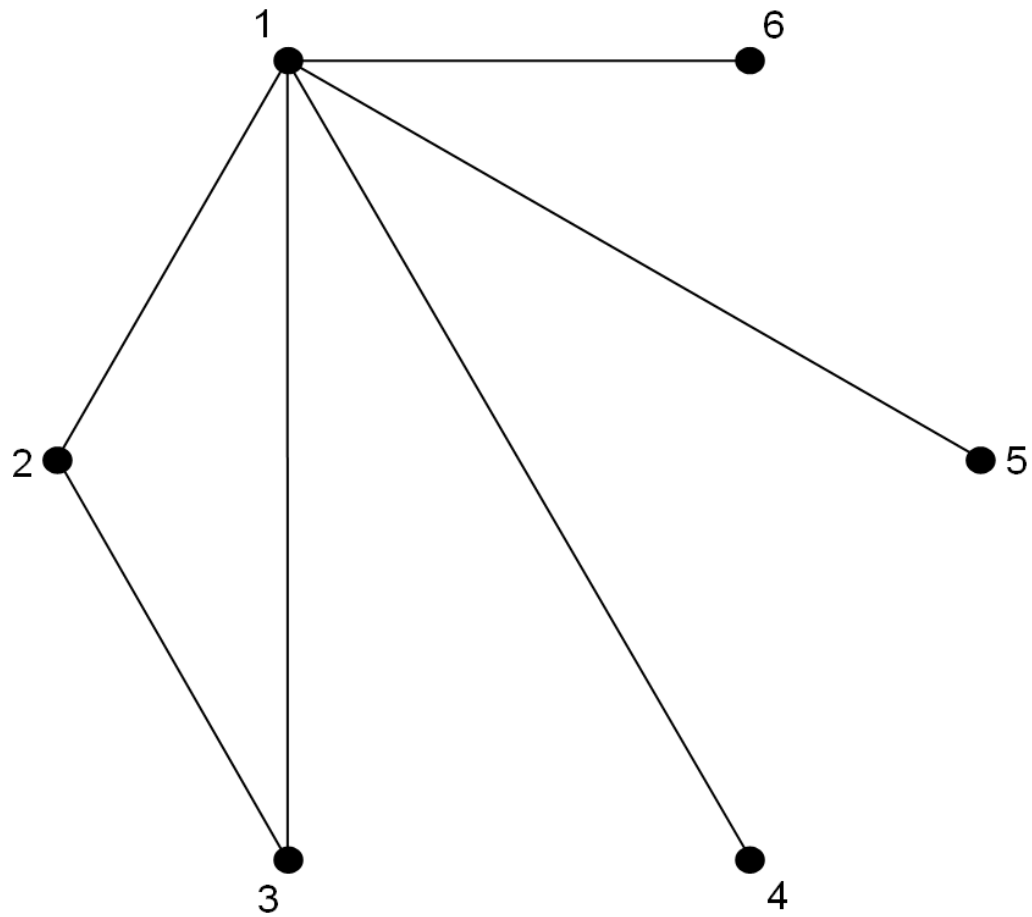
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 \\ 1/2 & 1 & & & \\ 1/5 & & 1 & & \\ 1/7 & & & 1 & \\ 1/9 & & & & 1 \\ & & & & & 1 \end{pmatrix}$$



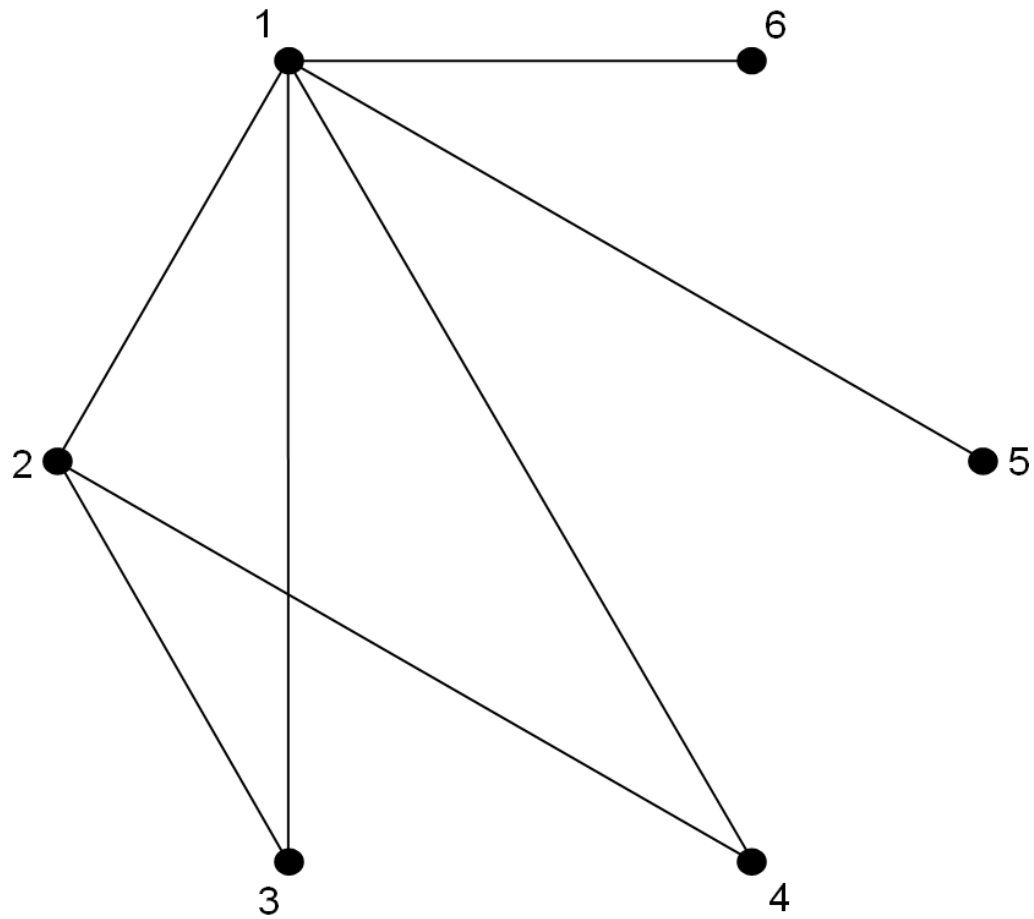
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & & & & \\ 1/5 & & 1 & & & \\ 1/7 & & & 1 & & \\ 1/9 & & & & 1 & \\ 1/9 & & & & & 1 \end{pmatrix}$$



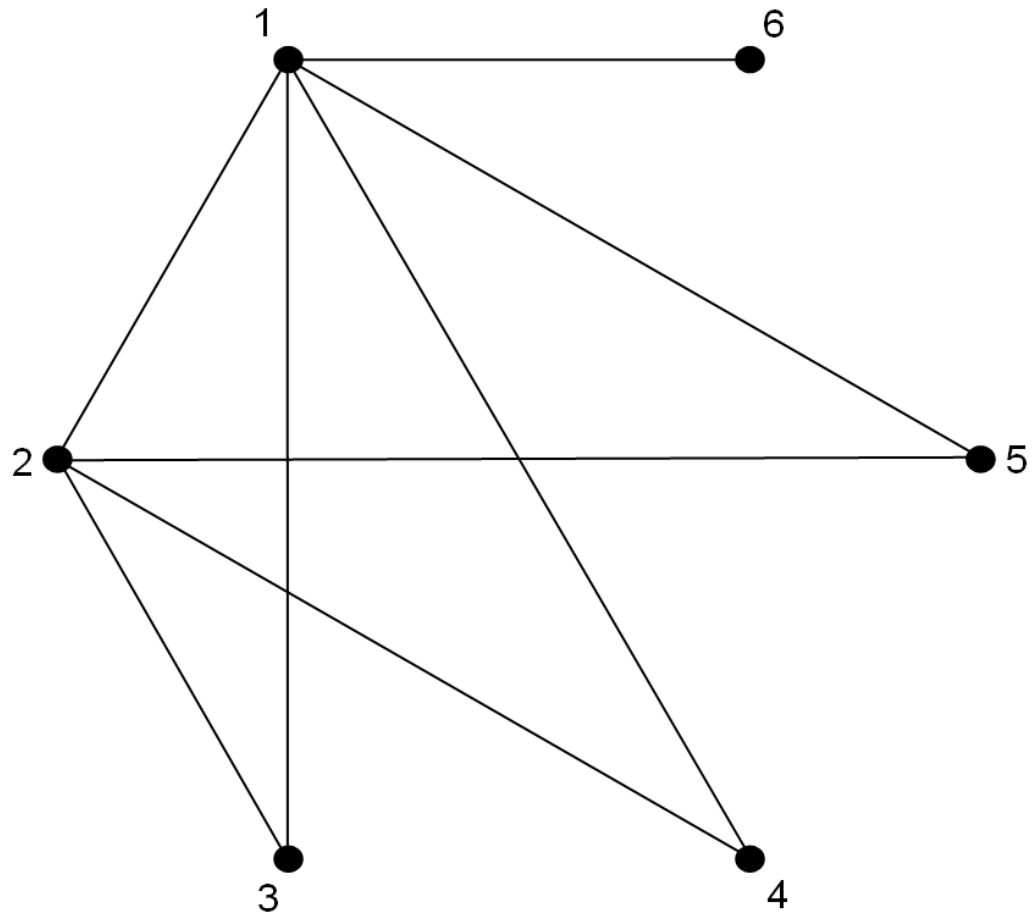
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & & & \\ 1/5 & 1/3 & 1 & & & \\ 1/7 & & & 1 & & \\ 1/9 & & & & 1 & \\ 1/9 & & & & & 1 \end{pmatrix}$$



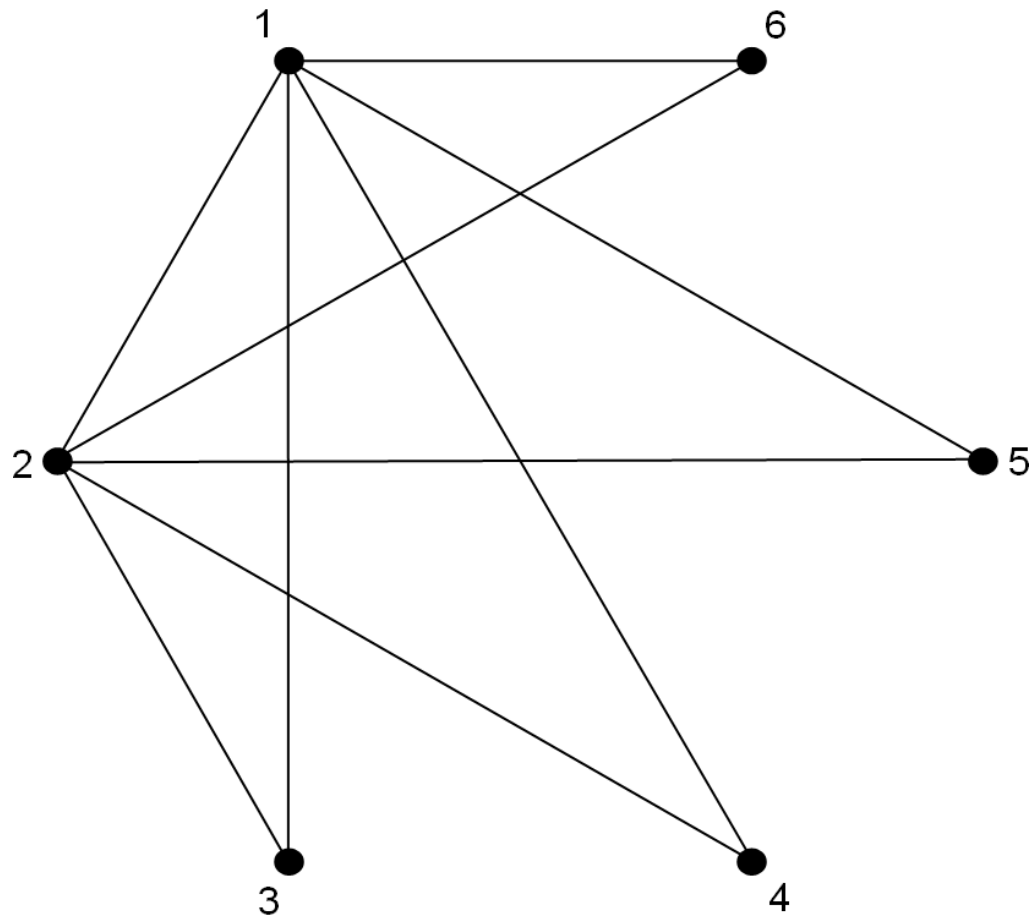
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & & \\ 1/5 & 1/3 & 1 & & & \\ 1/7 & 1/6 & & 1 & & \\ 1/9 & & & & 1 & \\ 1/9 & & & & & 1 \end{pmatrix}$$



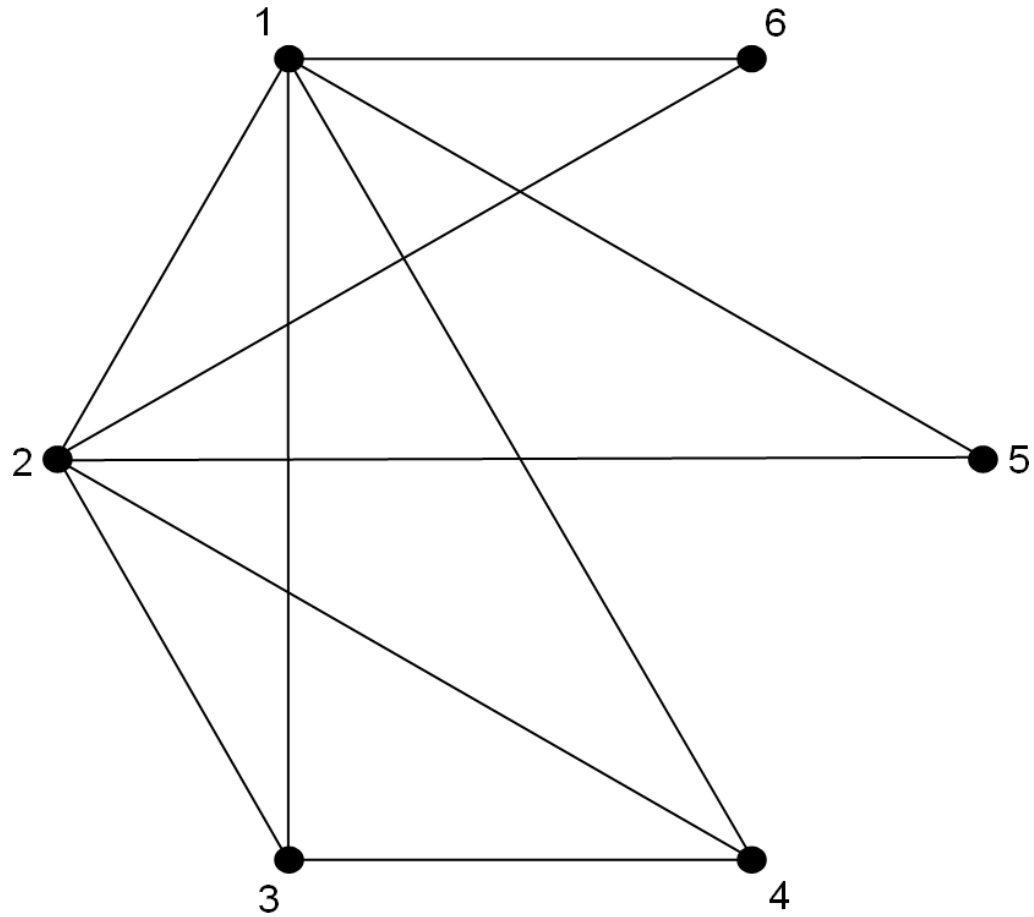
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & 8 & \\ 1/5 & 1/3 & 1 & & & \\ 1/7 & 1/6 & & 1 & & \\ 1/9 & 1/8 & & & 1 & \\ 1/9 & & & & & 1 \end{pmatrix}$$



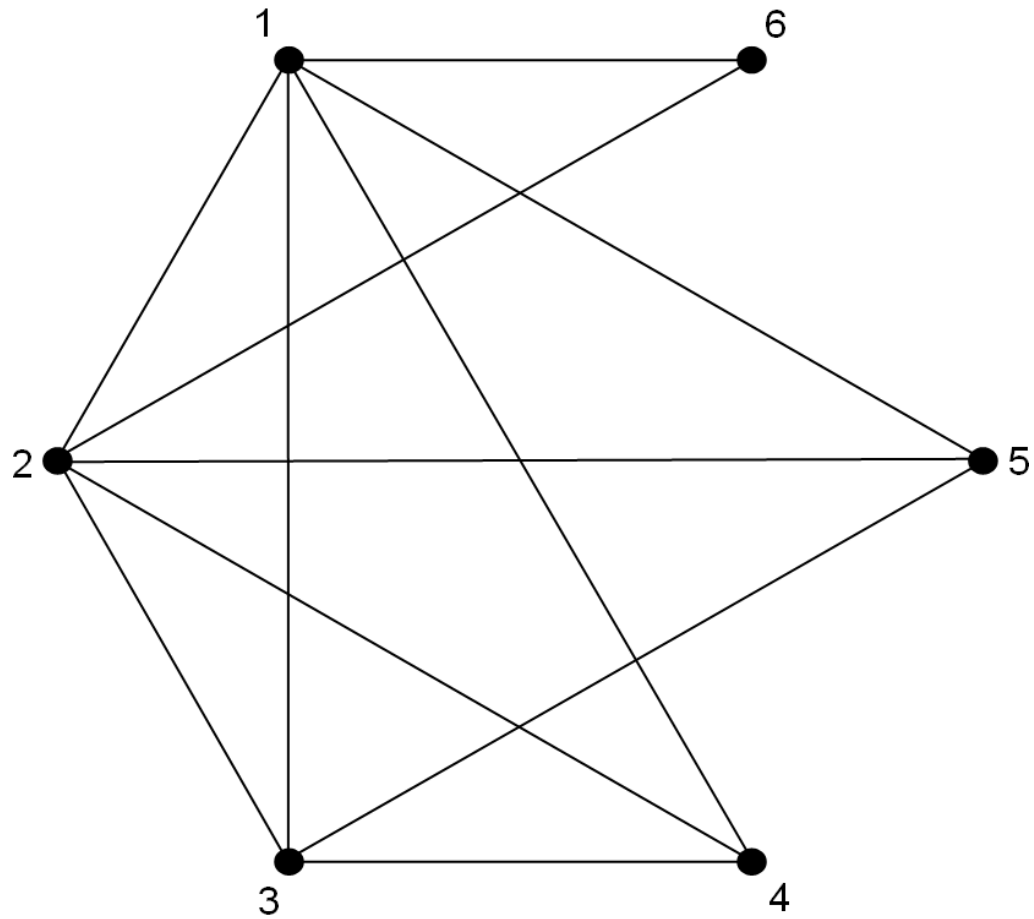
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & 8 & 9 \\ 1/5 & 1/3 & 1 & & & \\ 1/7 & 1/6 & & 1 & & \\ 1/9 & 1/8 & & & 1 & \\ 1/9 & 1/9 & & & & 1 \end{pmatrix}$$



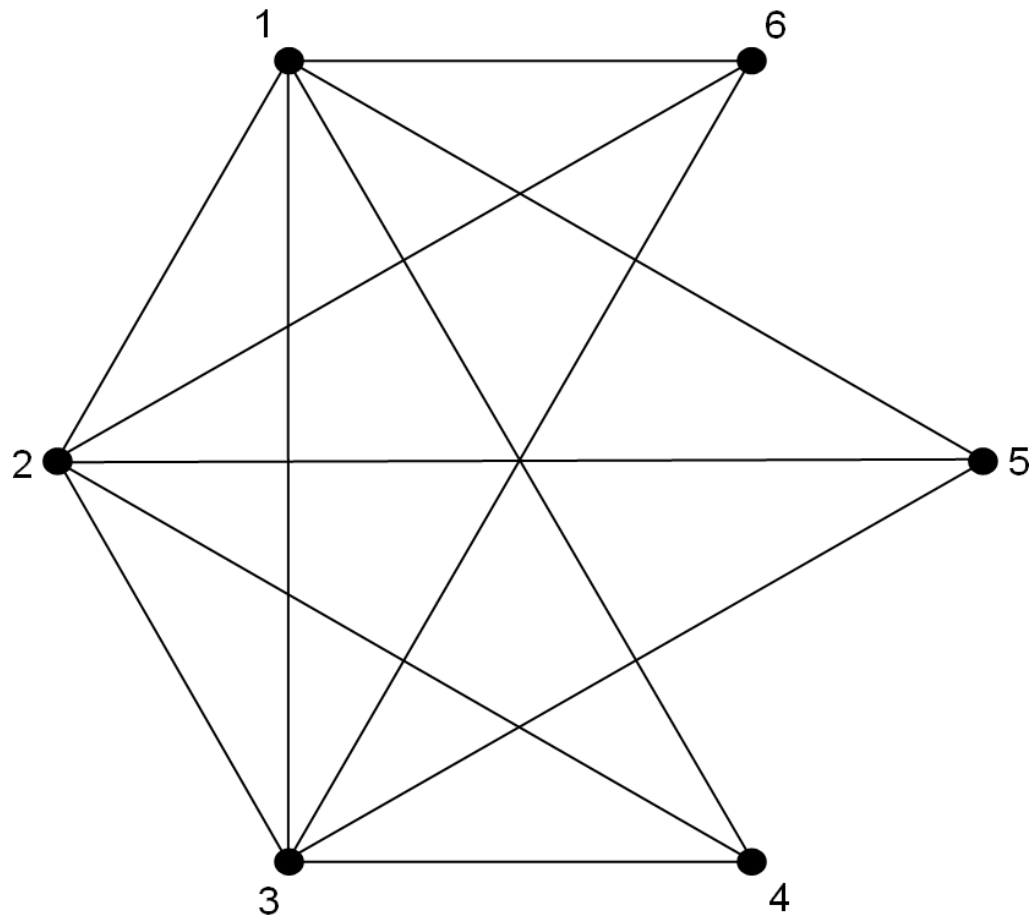
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & 8 & 9 \\ 1/5 & 1/3 & 1 & 4 & & \\ 1/7 & 1/6 & 1/4 & 1 & & \\ 1/9 & 1/8 & & & 1 & \\ 1/9 & 1/9 & & & & 1 \end{pmatrix}$$



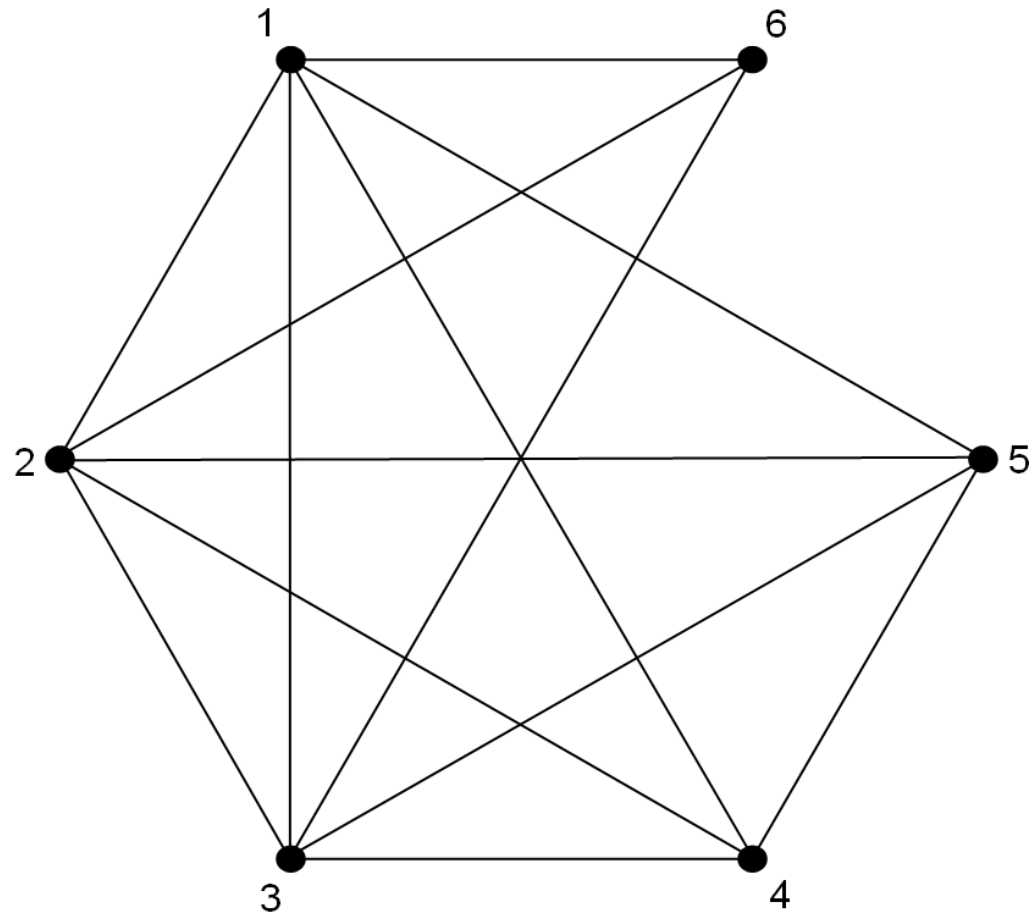
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & 8 & 9 \\ 1/5 & 1/3 & 1 & 4 & 6 & \\ 1/7 & 1/6 & 1/4 & 1 & & \\ 1/9 & 1/8 & 1/6 & & 1 & \\ 1/9 & 1/9 & & & & 1 \end{pmatrix}$$



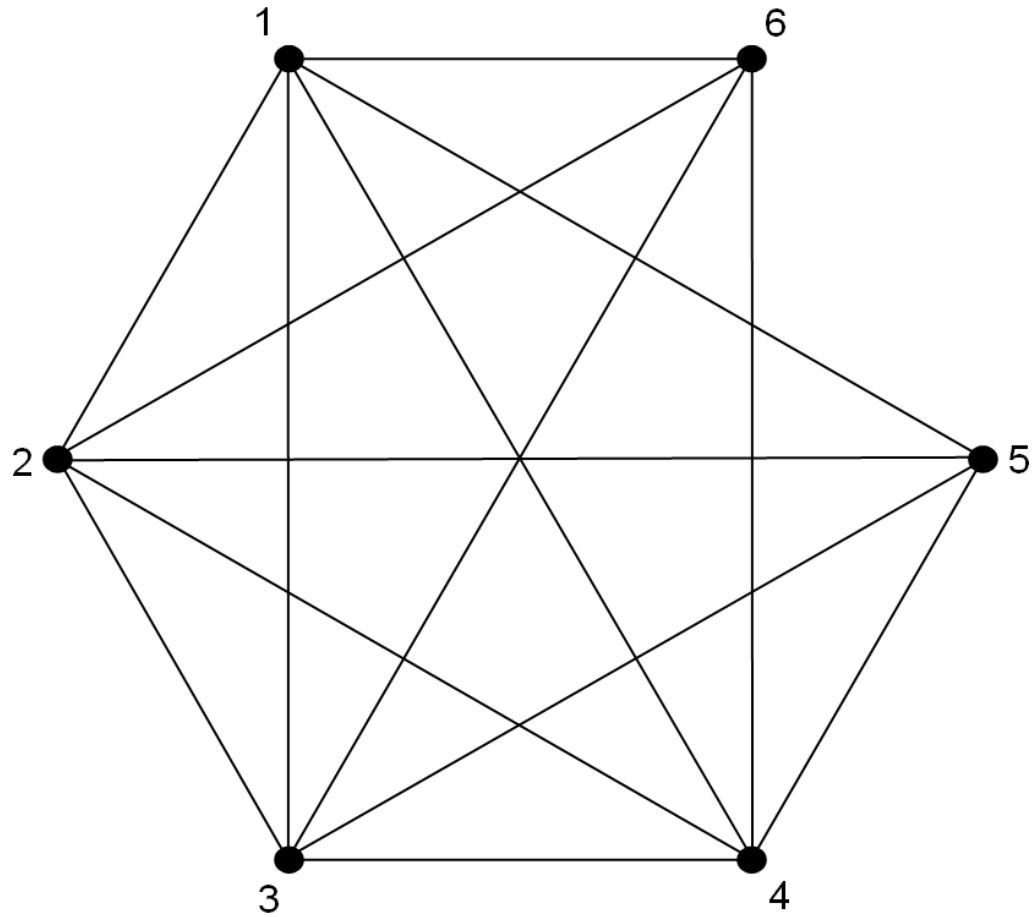
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & 8 & 9 \\ 1/5 & 1/3 & 1 & 4 & 6 & 7 \\ 1/7 & 1/6 & 1/4 & 1 & & \\ 1/9 & 1/8 & 1/6 & & 1 & \\ 1/9 & 1/9 & 1/7 & & & 1 \end{pmatrix}$$



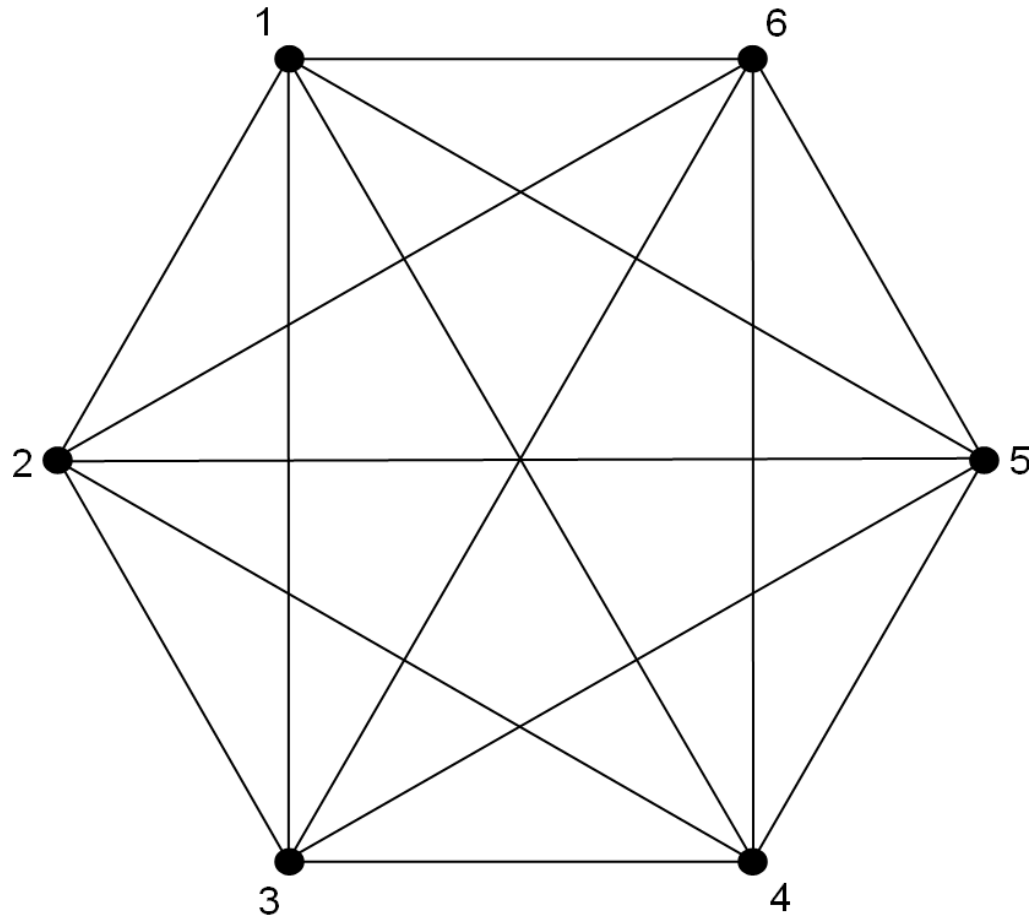
$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & 8 & 9 \\ 1/5 & 1/3 & 1 & 4 & 6 & 7 \\ 1/7 & 1/6 & 1/4 & 1 & 3 & 3 \\ 1/9 & 1/8 & 1/6 & 1/3 & 1 & 1 \\ 1/9 & 1/9 & 1/7 & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & 8 & 9 \\ 1/5 & 1/3 & 1 & 4 & 6 & 7 \\ 1/7 & 1/6 & 1/4 & 1 & 3 & 5 \\ 1/9 & 1/8 & 1/6 & 1/3 & 1 & \\ 1/9 & 1/9 & 1/7 & 1/5 & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 5 & 7 & 9 & 9 \\ 1/2 & 1 & 3 & 6 & 8 & 9 \\ 1/5 & 1/3 & 1 & 4 & 6 & 7 \\ 1/7 & 1/6 & 1/4 & 1 & 3 & 5 \\ 1/9 & 1/8 & 1/6 & 1/3 & 1 & 2 \\ 1/9 & 1/9 & 1/7 & 1/5 & 1/2 & 1 \end{pmatrix}$$



Saaty defined the inconsistency ratio as $CR = \frac{\lambda_{max} - n}{RI_n}$, where λ_{max} is the Perron eigenvalue of the complete pairwise comparison matrix given by the decision maker, and RI_n is defined as $\frac{\overline{\lambda_{max} - n}}{n-1}$, where $\overline{\lambda_{max}}$ is an average value of the Perron eigenvalues of randomly generated $n \times n$ pairwise comparison matrices.

It is well known that $\lambda_{max} \geq n$ and equals to n if and only if the matrix is consistent, i.e., the transitivity property holds. It follows from the definition that CR is a positive linear transformation of λ_{max} .

According to Saaty, larger value of CR indicates higher level of inconsistency and the 10%-rule ($CR \leq 0.10$) separates acceptable matrices from unacceptable ones.

Theorem (B., Fülöp, Rónyai, 2010): The optimal solution of the eigenvalue minimization problem

$$\min_{\mathbf{x} > \mathbf{0}} \lambda_{max}(\mathbf{A}(\mathbf{x})).$$

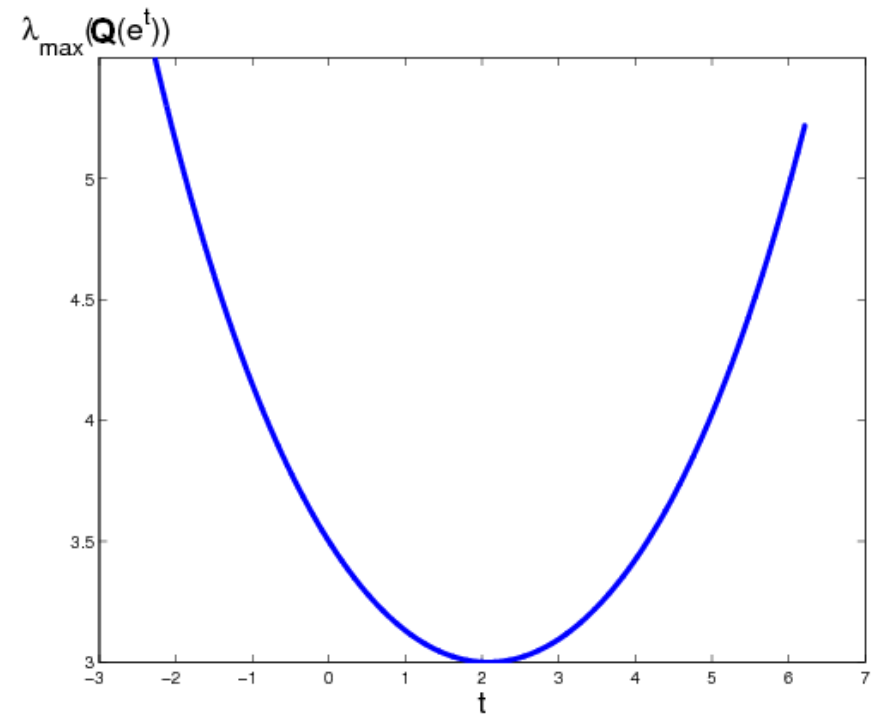
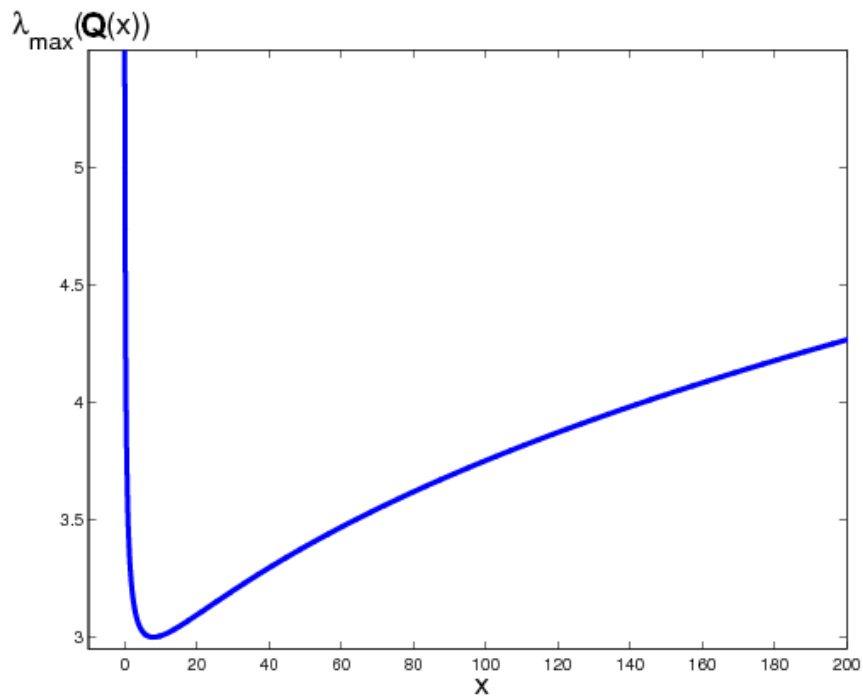
is unique if and only if the graph G corresponding to the incomplete pairwise comparison matrix is *connected*.

If graph G corresponding to the incomplete pairwise comparison matrix is connected, then by using the exponential parametrization $x_1 = e^{y_1}, x_2 = e^{y_2}, \dots, x_d = e^{y_d}$, the eigenvalue minimization problem is transformed into a strictly convex optimization problem.

Example

$$\mathbf{Q} = \begin{pmatrix} 1 & 2 & x \\ 1/2 & 1 & 4 \\ 1/x & 1/4 & 1 \end{pmatrix}.$$

$\lambda_{max}(\mathbf{Q}(x))$ and, by using the exponential scaling $x = e^t$, $\lambda_{max}(\mathbf{Q}(e^t))$ are plotted.



Algorithms for solving the eigenvalue minimization problem

$$\min_{\mathbf{x} > \mathbf{0}} \lambda_{max}(\mathbf{A}(\mathbf{x})).$$

- cyclic coordinates with Matlab's function *fminbnd*
- cyclic coordinates univariate Newton's method
- multivariate Newton's method

Method of cyclic coordinates

$$\mathbf{M}(\mathbf{x}) = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & x_1 & 5 & x_2 & 3 & x_3 & 1/7 \\ 1/3 & 1/x_1 & 1 & x_4 & 3 & x_5 & 6 & x_6 \\ 1/7 & 1/5 & 1/x_4 & 1 & x_7 & 1/4 & x_8 & 1/8 \\ 1/6 & 1/x_2 & 1/3 & 1/x_7 & 1 & x_9 & 1/5 & x_{10} \\ 1/6 & 1/3 & 1/x_5 & 4 & 1/x_9 & 1 & x_{11} & 1/6 \\ 3 & 1/x_3 & 1/6 & 1/x_8 & 5 & 1/x_{11} & 1 & x_{12} \\ 4 & 7 & 1/x_6 & 8 & 1/x_{10} & 6 & 1/x_{12} & 1 \end{pmatrix}$$

Method of cyclic coordinates

Let $x_i^{(k)}$ denote the value of x_i in the k -th step of the iteration, which has d (in the example, $d = 12$) substeps for each k .

For $k = 0$:

Let the initial points be equal to 1 for every variable:

$$x_i^{(0)} := 1 \quad (i = 1, 2, \dots, d).$$

while $\max_{i=1,2,\dots,d} \|x_i^k - x_i^{k-1}\| > T$

$x_i^{(k)} :=$

$\arg \min_{x_i} \lambda_{max}(\mathbf{M}(x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k-1)}, \dots, x_d^{(k-1)}))$, $i =$

$1, 2, \dots, d$

next k

end while

Method of cyclic coordinates

Focus on $\min_{x_i} \lambda_{max}(\mathbf{M}(x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k-1)}, \dots, x_d^{(k-1)}))$

Matlab's function *fminbnd* solves is directly and fast.

Univariate Newton's method can be also applied.

$$\min_{x>0} \lambda_{max}(\mathbf{A}(x))$$

Let $x = e^t$ and $L(t) = \lambda_{max}(e^t)$.

$$t_{n+1} = t_n - \frac{L'(t_n)}{L''(t_n)} = t_n - \frac{\frac{\partial \lambda_{max}(x)}{\partial x}}{\frac{\partial^2 \lambda_{max}(x)}{(\partial x)^2} \cdot e^{t_n} + \frac{\partial \lambda_{max}(x)}{\partial x}}.$$

By Harker, formal derivatives $\frac{\partial \lambda_{max}(x)}{\partial x}$ and $\frac{\partial^2 \lambda_{max}(x)}{(\partial x)^2}$ are known.

Harker's formula for the first derivative $\frac{\partial \lambda_{max}(x)}{\partial x}$

$$\left(\frac{\partial \lambda_{max}(A)}{\partial a_{ij}} \Big|_{i > j} \right) = \left([y(A)_i x(A)_j] - \frac{[y(A)_j x(A)_i]}{[a_{ij}]^2} \right)$$

where vectors $x(A), y(A)$ are the right-hand side and left-hand side eigenvectors of A , respectively.

Normalization $y(A)^T x(A) = 1$ is applied.

Harker's formula for the second derivative

$$\begin{aligned}
 \frac{\partial^2 \lambda_{max}(A)}{\partial a_{ij} \partial a_{kl}} = & (x(A)y(A)^T)_{li} Q_{jk}^+ + (x(A)y(A)^T)_{jk} Q_{li}^+ \\
 & \frac{(x(A)y(A)^T)_{ki} Q_{jl}^+ + (x(A)y(A)^T)_{jl} Q_{ki}^+}{[a_{kl}]^2} \\
 & \frac{(x(A)y(A)^T)_{lj} Q_{ik}^+ + (x(A)y(A)^T)_{ik} Q_{lj}^+}{[a_{ij}]^2} \\
 & + \frac{(x(A)y(A)^T)_{kl} Q_{il}^+ + (x(A)y(A)^T)_{il} Q_{kj}^+}{[a_{ij}]^2 [a_{kl}]^2}
 \end{aligned}$$

if $i \neq k$ or $j \neq l$,

Harker's formula for the second derivative

$$\begin{aligned} \frac{\partial^2 \lambda_{max}(A)}{\partial a_{ij} \partial a_{kl}} = & \frac{2(x(A)y(A)^T)_{ij}}{[a_{ij}]^3} + 2(x(A)y(A)^T)_{ji}Q_{ii}^+ \\ & - 2 \frac{(x(A)y(A)^T)_{ii}Q_{jj}^+ + (x(A)y(A)^T)_{jj}Q_{ii}^+}{[a_{ij}]^2} \\ & + 2 \frac{(x(A)y(A)^T)_{ij}Q_{ij}^+}{[a_{ij}]^4} \end{aligned}$$

if $i = k$ and $j = l$, where $Q = \lambda_{max}(A)I - A$ and Q^+ denotes the Moore-Penrose pseudoinverse of Q .

Multivariate Newton's method

$$\underline{t}_{n+1} = \underline{t}_n - [HL(\underline{t}_n)]^{-1} \nabla L(\underline{t}_n).$$

Computational results:

- All methods mentioned above are fast enough for typical matrices in multi-attribute decision making;
- in the talk's example (8×8 matrix, 12 variables, 20 cycles with *fminbnd*: 0.3 seconds, 1-variable Newton: 5.3 seconds);

Koczkodaj's CM index of inconsistency

$$\begin{pmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{pmatrix}$$

Koczkodaj defined the inconsistency index as

$$CM(a, b, c) = \min \left\{ \frac{1}{a} \left| a - \frac{b}{c} \right|, \frac{1}{b} |b - ac|, \frac{1}{c} \left| c - \frac{b}{a} \right| \right\}.$$

Koczkodaj's CM index of inconsistency

$$CM(a, b, c) = \min \left\{ \frac{1}{a} \left| a - \frac{b}{c} \right|, \frac{1}{b} |b - ac|, \frac{1}{c} \left| c - \frac{b}{a} \right| \right\}.$$

Duszak and Koczkodaj extended this definition for a general $n \times n$ reciprocal matrix \mathbf{A} as the maximum of $CM(a, b, c)$ for all triads (a, b, c) , i.e., 3×3 submatrices which are themselves pairwise comparison matrices, in \mathbf{A} :

$$CM(\mathbf{A}) = \max \{ CM(a_{ij}, a_{ik}, a_{jk}) \mid 1 \leq i < j < k \leq n \}.$$

The extension of index CM for incomplete case

Proposition (B., Fülöp, Koczkodaj, 2011): The optimal solution of the optimization problem

$$\min_{\mathbf{x} > \mathbf{0}} CM(\mathbf{A}(\mathbf{x})).$$

can be transformed into an LP problem.

Applications of the results:

- A generalization of the Eigenvector Method for the incomplete case
- CR and CM inconsistency indices can be computed during the filling in process, as soon as a connected graph is given
- User may get an automatic warning in case of misprints, detected as a high jump in CR or CM inconsistency.

Questions:

- How many comparisons are needed, if less than $n(n - 1)/2$?
- Rules and thresholds for warning the user?
- Ordinal inconsistency has been investigated by Gass (1998) and Kéri (2010) for the complete case. Incomplete versions are of future research.

References 1/4

Ábele-Nagy, K. [2010]: Incomplete pairwise comparison matrices in multi-attribute decision making, *Master's Thesis*, Eötvös Loránd University.

Bozóki, S., Fülöp, Koczkodaj, W.W. [2011]: An LP-based inconsistency monitoring of pairwise comparison matrices, *Mathematical and Computer Modelling*, **54**(1-2), pp. 789-793.

Bozóki, S., Fülöp, J., Rónyai, L. [2010]: On optimal completions of incomplete pairwise comparison matrices, *Mathematical and Computer Modelling*, **52**, pp. 318–333.

Duszak, Z., Koczkodaj, W.W. [1994]: Generalization of a new definition of consistency for pairwise comparisons, *Information Processing Letters*, **52**, pp. 273–276.

References 2/4

Gass, S.I. [1998]: Tournaments, transitivity and pairwise comparison. *Journal of the Operational Research Society*, **49**, pp. 616–624.

Harker, P.T. [1987]: Incomplete pairwise comparisons in the analytic hierarchy process. *Mathematical Modelling*, **9**(11), pp. 837–848.

Harker, P.T. [1987]: Derivatives of the Perron root of a positive reciprocal matrix: with application to the Analytic Hierarchy Process. *Applied Mathematics and Computation*, **22**, pp. 217–232.

Kéri, G. [2010]: On qualitatively consistent, transitive and contradictory judgment matrices emerging from multiattribute decision procedures. *Central European Journal of Operations Research*, **19**(2), pp. 215–224.

References 3/4

Koczkodaj, W.W. [1993]: A New Definition of Consistency of Pairwise Comparisons, *Mathematical and Computer Modelling*, **18**(7), pp. 79–84.

Saaty, T.L. [1977]: A scaling method for priorities in hierarchical structures, *Journal of Mathematical Psychology*, **15**, pp. 234–281.

Saaty, T.L. [1980]: The analytic hierarchy process, *McGraw-Hill*, New York.

Shiraishi, S., Obata, T., Daigo, M. [1998]: Properties of a positive reciprocal matrix and their application to AHP. *Journal of the Operations Research Society of Japan*, **41**(3), pp. 404–414.

References 4/4

Shiraishi, S., Obata, T. [2002]: On a maximization problem arising from a positive reciprocal matrix in AHP. *Bulletin of Informatics and Cybernetics*, **34**(2), pp. 91–96.

Thank you for attention.

bozoki@sztaki.hu

<http://www.sztaki.hu/~bozoki>