

On optimal completions of incomplete pairwise comparison matrices

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Abstract

An important variant of a key problem for multi-attribute decision making is considered. We study the extension of the pairwise comparison matrix to the case when only partial information is available: for some pairs no comparison is given. It is natural to define the inconsistency of a partially filled matrix as the inconsistency of its best, completely filled completion. We study here the uniqueness problem of the best completion for two weighting methods, the Eigenvector Method and the Logarithmic Least Squares Method. In both settings we obtain the same simple graph theoretic characterization of the uniqueness. The optimal completion will be unique if and only if the graph associated with the partially defined matrix is connected. Some numerical experiences are discussed at the end of the paper.

Keywords: Multiple criteria analysis, Incomplete pairwise comparison matrix, Perron eigenvalue, Convex programming

1 Introduction

Pairwise comparisons are often used in reflecting cardinal preferences, especially in multi-attribute decision making, for computing the weights of criteria or evaluating the alternatives with respect to a criterion. It is assumed that decision makers do not know the weights of criteria (values of the alternatives) explicitly. However, they are able to compare any pairs of

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the criteria (alternatives). Pairwise comparison matrix [26, 27] is defined as follows. Given n objects (criteria or alternatives) to compare, the pairwise comparison matrix is $\mathbf{A} = [a_{ij}]_{i,j=1,\dots,n}$, where a_{ij} is the numerical answer given by the decision maker for the question 'How many times Criterion i is more important than Criterion j ?', or, analogously, 'How many times Alternative i is better or preferred than Alternative j with respect to a given criterion?'. A pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix}, \quad (1)$$

is positive and reciprocal, i.e.,

$$\begin{aligned} a_{ij} &> 0, \\ a_{ij} &= \frac{1}{a_{ji}}, \end{aligned} \quad (2)$$

for $i, j = 1, \dots, n$.

The problem is to determine the positive weight vector $\mathbf{w} = (w_1, w_2, \dots, w_n)^T \in \mathbb{R}_+^n$, (\mathbb{R}_+^n denotes the positive orthant of the n -dimensional Euclidean space), such that the appropriate ratios of the components of \mathbf{w} reflect all the pairwise comparisons, given by the decision maker, *as well as possible*. In fact, there are several mathematical models for the objective 'as well as possible'. A comparative study of weighting methods is done by Golany and Kress [10] and a more recent one by Ishizaka and Lusti [16]. In the present paper, two well-known methods, the Eigenvector Method (*EM*) [26, 27] and the Logarithmic Least Squares Method (*LLSM*) [6, 7] are considered.

In the **Eigenvector Method** (*EM*) the approximation \mathbf{w}^{EM} of \mathbf{w} is defined by

$$\mathbf{A}\mathbf{w}^{EM} = \lambda_{max}\mathbf{w}^{EM},$$

where λ_{max} denotes the maximal eigenvalue, also known as Perron eigenvalue, of \mathbf{A} and \mathbf{w}^{EM} denotes the the right-hand side eigenvector of \mathbf{A} corresponding to λ_{max} . By Perron's theorem, \mathbf{w}^{EM} is positive and unique up to a scalar multiplication [23]. The most often used normalization is $\sum_{i=1}^n w_i^{EM} = 1$.

The **Logarithmic Least Squares Method (LLSM)** gives \mathbf{w}^{LLSM} as the optimal solution of

$$\begin{aligned} \min \sum_{i=1}^n \sum_{j=1}^n \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2 \\ \sum_{i=1}^n w_i = 1, \\ w_i > 0, \quad i = 1, 2, \dots, n. \end{aligned} \quad (3)$$

The optimization problem (3) is known to be solvable, and has a unique optimal solution, which can be explicitly computed by taking the geometric means of rows' elements [7, 6].

Both the Eigenvector and Logarithmic Least Squares Methods are defined for complete pairwise comparison matrices, that is, when all the n^2 elements of the matrix are known.

Both complete and incomplete pairwise comparisons may be modelled not only in multiplicative form (1)-(2) but also in additive way as in [8, 9, 33, 34]. Additive models of cardinal preferences are not included in the scope of this investigation. However, some of the main similarities and differences between multiplicative and additive models are indicated.

In the paper, we consider an incomplete version of the models discussed above (Section 2). We assume that our expert has given estimates a_{ij} only for a subset of the pairs (i, j) . This may indeed be the case in practical situations. If the number n of objects is large, then it may be a prohibitively large task to give as many as $\binom{n}{2}$ thoughtful estimates. Also, it may be the case that the expert agent is less certain about ranking certain pairs (i', j') than others, and is willing to give estimates only for those of the pairs, where s/he is confident enough as to the quality of the estimate.

Harker proposed a method for determining the weights from incomplete matrices, based on linear algebraic equations that could be interpreted not only in the complete case [12, 13]. Kwiesielewicz [20] considered the Logarithmic Least Squares Method for incomplete matrices and proposed a weighting method based on the generalized pseudoinverse matrices.

A pairwise comparison matrix in (1) is called *consistent* if the transitivity $a_{ij}a_{jk} = a_{ik}$ holds for all indices $i, j, k = 1, 2, \dots, n$. Otherwise, the matrix

is *inconsistent*. However, there are different levels of inconsistency, some of which are acceptable in solving real decision problems, some are not. Measuring, or, at least, indexing inconsistency is still a topic of recent research [4].

Saaty [26, 27] defined the inconsistency ratio as $CR = \frac{\lambda_{max} - n}{RI_n}$, where λ_{max} is the Perron eigenvalue of the complete pairwise comparison matrix given by the decision maker, and RI_n is defined as $\frac{\overline{\lambda_{max}} - n}{n-1}$, where $\overline{\lambda_{max}}$ is an average value of the Perron eigenvalues of randomly generated $n \times n$ pairwise comparison matrices. It is well known that $\lambda_{max} \geq n$ and equals to n if and only if the matrix is consistent, i.e., the transitivity property holds. It follows from the definition that CR is a positive linear transformation of λ_{max} . According to Saaty, larger value of CR indicates higher level of inconsistency and the 10%-rule ($CR \leq 0.10$) separates acceptable matrices from unacceptable ones.

Based on the idea above, Shiraishi, Obata and Daigo [28, 29] considered the eigenvalue optimization problems as follows. In case of one missing element, denoted by x , the $\lambda_{max}(\mathbf{A}(x))$ to be minimized:

$$\min_{x>0} \lambda_{max}(\mathbf{A}(x)).$$

In case of more than one missing elements, arranged in vector \mathbf{x} , the aim is to solve

$$\min_{\mathbf{x}>\mathbf{0}} \lambda_{max}(\mathbf{A}(\mathbf{x})). \quad (4)$$

For a given incomplete matrix \mathbf{A} , the notations $\lambda_{max}(\mathbf{A}(\mathbf{x}))$ and $\lambda_{max}(\mathbf{x})$ are both used with the same meaning.

Pairwise comparisons may be represented not only by numbers arranged in a matrix but also by directed and undirected graphs with weighted edges [17]. A natural correspondence is presented between pairwise comparison matrices and edge weighted graphs.

A generalization of the Eigenvector Method for the incomplete case is introduced in Section 3. A basic concept of *EM* is that λ_{max} is strongly related to the level of inconsistency: larger λ_{max} value indicates that the pairwise comparison matrix is more inconsistent. Based on this idea, the aim is to minimize the maximal eigenvalue among the complete positive reciprocal matrices extending the partial matrix specified by the agent. Here the minimum is taken over all possible positive reciprocal completions of the partial matrix. Based on Kingman's theorem [18], Aupetit and Genest pointed out that when all entries of \mathbf{A} are held constant except a_{ij} and

$a_{ji} = \frac{1}{a_{ij}}$ for fixed $i \neq j$, then $\lambda_{max}(\mathbf{A})$ is a logconvex (hence convex) function of $\log a_{ij}$. We show that this result can be easily extended to the case when several entries of \mathbf{A} are considered simultaneously as variables. This makes it possible to reformulate the maximal eigenvalue minimization problem (4) as an unconstrained convex minimization problem.

The question of uniqueness of the optimal solution occurs naturally. The first main result of the paper is solving the eigenvector optimization problem (4) and giving necessary and sufficient conditions for the uniqueness.

In Section 4, a distance minimizing method, the Logarithmic Least Squares Method (*LLSM*) is considered. The extension of (3) to the incomplete case appears to be straightforward. In calculating the optimal vector \mathbf{w} we consider in the objective function the terms $\left(\log a_{ij} - \log \frac{w_i}{w_j}\right)^2$ only for those pairs (i, j) for which there exists an estimate a_{ij} .

The second main result of the paper is solving and discussing the incomplete *LLSM* problem. Analogously to *EM*, the same necessary and sufficient condition for the existence and uniqueness of the optimal solution is provided. In both settings the connectedness of the associated graph characterizes the unique solvability of the problem.

Connectedness is a natural and elementary necessary condition but sufficiency is not trivial. Harker's approximation of a missing comparison between items i and j is based on the product of known matrix elements which connect i to j [13]. This idea is also applied in our proof of Theorem 2. Fedrizzi and Giove apply connected subgraphs for indicating the dependence or independence of missing variables [9](Section 3., pp. 308-309). However, in the paper, the connectedness of known elements is analyzed.

The third main result of the paper is a new algorithm proposed in Section 5, based on the results of Section 3 for finding the best completion according to the *EM* model (i.e., with minimal λ_{max}) of an incomplete pairwise comparison matrix. The well-known method of cyclic coordinates [3] is used.

In Section 6, numerical examples are presented for both incomplete *EM* and *LLSM* models. It is also shown that van Uden's rule ([31],[21]) provides a very good approximation for the missing elements and may be used as a starting point for the optimization algorithm of Section 5.

All the results of the paper hold for arbitrary ratio scales including but not restricted to $\{1/9, \dots, 1/2, 1, 2, \dots, 9\}$ proposed by Saaty [26, 27].

The conclusions of Section 7 ponder the questions of applicability from decision theoretical points of view.

2 Incomplete pairwise comparison matrices and graph representation

2.1 Incomplete pairwise comparison matrices

Let us assume that our pairwise comparison matrices are not completely given. It may happen that one or more elements are not given by the decision maker for various reasons. As we have pointed out in the Introduction, there may be several realistic reasons for this to happen.

Incomplete pairwise comparison matrices were defined by Harker [12, 13] and investigated in [5, 20, 21, 28, 28, 30, 31]. Additive models are analyzed in [8, 9, 33, 34]. The reference list of Fedrizzi and Giove [9] is offered for more models, both multiplicative and additive.

Incomplete pairwise comparison matrix is very similar to the form (1) but one or more elements, denoted here by $*$, are not given:

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & * & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & * \\ * & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & * & 1/a_{3n} & \dots & 1 \end{pmatrix}. \quad (5)$$

Most of the linear algebraic concepts, tools and formulas are defined for complete matrices rather than for incomplete ones. We introduce variables $x_1, x_2, \dots, x_d \in \mathbb{R}_+$ for the missing elements in the upper triangular part of \mathbf{A} . Their reciprocals, $1/x_1, 1/x_2, \dots, 1/x_d$ are written in the lower triangular part of \mathbf{A} as in (6). The total number of missing elements in matrix \mathbf{A} is $2d$. Let

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}(x_1, x_2, \dots, x_d) = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & x_d \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/x_d & 1/a_{3n} & \dots & 1 \end{pmatrix}, \quad (6)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_d)^T \in \mathbb{R}_+^d$. The form (6) is also called incomplete pairwise comparison matrix. However, it will be useful to consider them as a class of (complete) pairwise comparison matrices as realizations of \mathbf{A} , generated by all the values of $\mathbf{x} \in \mathbb{R}_+^d$. Our notation involving variables corresponds to the view that an incomplete pairwise comparison matrix \mathbf{A} is actually the collection of all fully specified comparison matrices which are

identical to \mathbf{A} at the previously specified entries.

From decision theoretical and practical points of view, the really important and exciting questions are: how to estimate weights and the level of inconsistency based on the known elements rather than somehow obtain possible, assumed, computed or generated values of the missing entries. Nevertheless, in many cases, 'optimal' values of \mathbf{x} resulted in by an algorithm may be informative as well.

2.2 Graph representation

Assume that the decision maker is asked to compare the relative importance of n criteria and s/he is filling in the pairwise comparison matrix. In each comparison, a direct relation is defined between two criteria, namely, the estimated ratio of their weights. However, two criteria, not compared yet, consequently, having no direct relation, can be in indirect relation, through further criteria and direct relations. It is a natural idea to associate graph structures to (in)complete pairwise comparison matrices.

Given an (in)complete pairwise comparison matrix \mathbf{A} of size $n \times n$, two graphs, G and \vec{G} are defined as follows:

$G := (V, E)$, where $V = \{1, 2, \dots, n\}$, the vertices correspond to the objects to compare and $E = \{e(i, j) \mid a_{ij} \text{ (and } a_{ji}) \text{ is given and } i \neq j\}$, the undirected edges correspond to the matrix elements. There are no edges corresponding to the missing elements in the matrix. G is an undirected graph.

$\vec{G} := (V, \vec{E})$, where $\vec{E} = \{e(i, j) \mid a_{ij} \text{ is given and } i \neq j\} \cup \{e(i, i) \mid i = 1, 2, \dots, n\}$, the directed edges correspond to the matrix elements. There are no edges corresponding to the missing elements in the matrix. \vec{G} is a directed graph, which can be obtained from G by orienting the edges of G in both ways. Moreover, we add loops to the vertices.

Example 1. Let \mathbf{C} be a 6×6 incomplete pairwise comparison matrix as follows:

$$\mathbf{C} = \begin{pmatrix} 1 & a_{12} & a_{13} & * & * & a_{16} \\ a_{21} & 1 & a_{23} & * & * & * \\ a_{31} & a_{32} & 1 & a_{34} & a_{35} & a_{36} \\ * & * & a_{43} & 1 & * & * \\ * & * & a_{53} & * & 1 & a_{56} \\ a_{61} & * & a_{63} & * & a_{65} & 1 \end{pmatrix}.$$

Then, the corresponding graphs G and \vec{G} are presented in Figures 1 and 2.

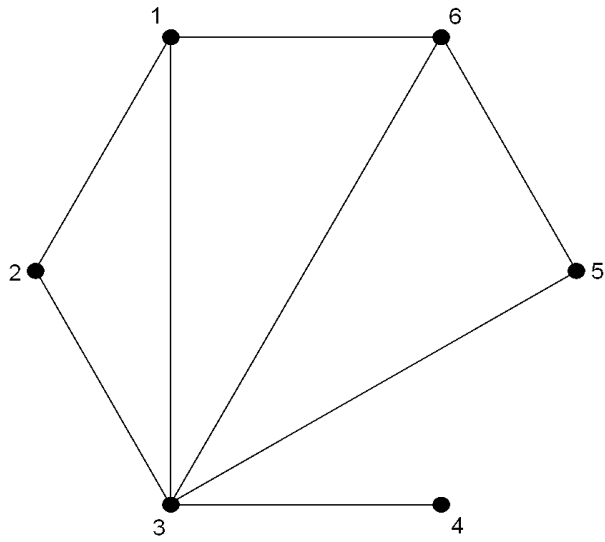


Figure 1. The undirected graph G corresponding to matrix \mathbf{C}

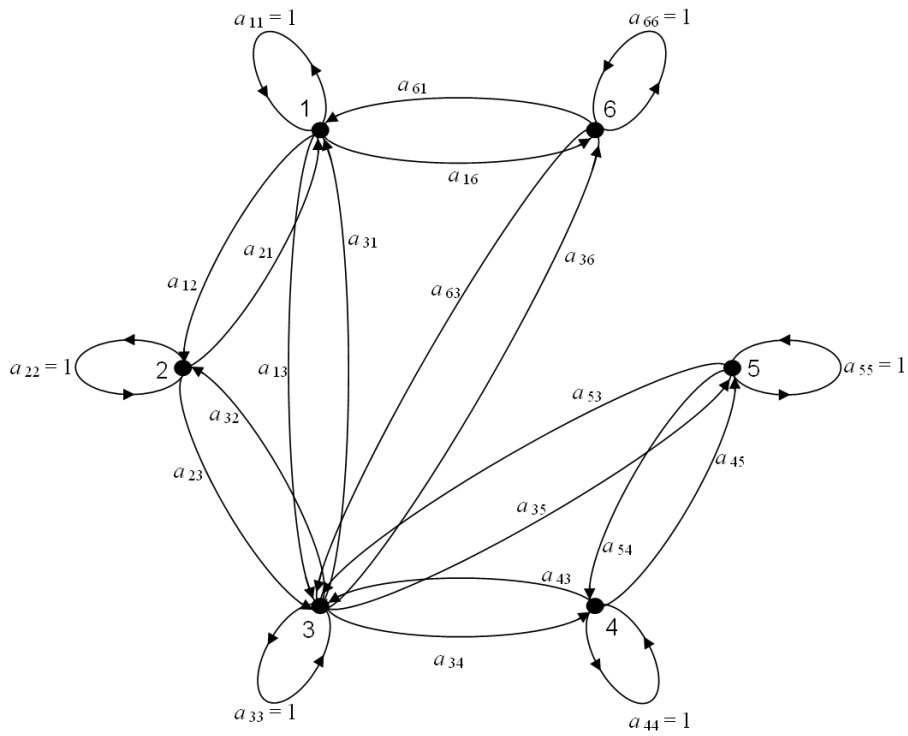


Figure 2. The directed graph \vec{G} corresponding to matrix \mathbf{C}

Remark 1. In the case of a complete matrix, G is the complete undirected graph of n vertices and \vec{G} is the complete directed graph of n vertices, the edges of which are oriented in both ways and loops are added at the vertices.

3 The eigenvector method for incomplete matrices

In this section, the eigenvalue optimization problem (4) is discussed. Starting from an incomplete pairwise comparison matrix in form (6), we show that problem (4) can be transformed into a convex optimization problem, moreover, under a natural condition, into a strictly convex optimization problem. This latter statement is our main result here.

Let us parameterize the entries of the (complete or incomplete) pairwise comparison matrix $\mathbf{A}(\mathbf{x}) = \mathbf{A}(x_1, x_2, \dots, x_d)$ of form (6) as $x_i = e^{y_i}$, ($i = 1, 2, \dots, d$). This way we obtain a matrix

$$\mathbf{A}(\mathbf{x}) = \mathbf{B}(\mathbf{y}) = \mathbf{B}(y_1, y_2, \dots, y_d). \quad (7)$$

Definition 1. Let $C \subseteq \mathbb{R}^k$ be a convex set and $f : C \rightarrow \mathbb{R}_+$. A function f is called *logconvex* (or *superconvex*), if $\log f : C \rightarrow \mathbb{R}$ is a convex function.

It is easy to see that a logconvex function is convex as well.

Proposition 1. For the parametrized matrix $\mathbf{B}(\mathbf{y})$ from (7) the Perron eigenvalue $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ is a logconvex (hence convex) function of \mathbf{y} .

Proposition 1 has been proved by Aupetit and Genest [2], for the case when $d = 1$. The proof is based on Kingman's theorem as follows.

Theorem 1. (Kingman [18]): If the elements of matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, denoted by $a_{ij}(t)$, ($i, j = 1, 2, \dots, n$) are logconvex functions of t , then $\lambda_{max}(\mathbf{A}(t))$ is logconvex (hence convex).

Proposition 1 for general d can also be proved by applying Theorem 1. It is enough to show the logconvexity along the lines in the space \mathbf{y} . The pairwise comparison matrices along any line in the \mathbf{y} space can be written as

$$\mathbf{B}(t) = [e^{c_{ij}t + d_{ij}}]_{i,j=1,\dots,n}, \quad (8)$$

where t is a scalar variable, and $c_{ij}, d_{ij} \in \mathbb{R}$. Note that a matrix of this form is a pairwise comparison matrix for every value of t if and only if $c_{ii} = d_{ii} = 0$; and $c_{ij} = -c_{ji}$, $d_{ij} = -d_{ji}$ hold for every i, j . Note also, that if the value a_{ij} is known, then we have $c_{ij} = 0$, and $d_{ij} =$

$\log a_{ij}$. Using the parametrization of the line, Proposition 1 follows immediately from Theorem 1. However, for the reader's convenience, and since some ideas of the proof will be used later, we outline a simple direct proof of Proposition 1.

Proof of Proposition 1: It suffices to show logconvexity along lines in the \mathbf{y} space. It is sufficient to prove that $\lambda_{max}(\mathbf{B}(t))$ is a logconvex function of the real variable t .

It is known (see, e.g., in [15], take the trace in Theorem 8.2.8) that for a strictly positive matrix \mathbf{A} one has

$$\lambda_{max}(\mathbf{A}) = \lim_{m \rightarrow \infty} \sqrt[m]{Tr(\mathbf{A}^m)}.$$

Write

$$f(t) = f_m(t) = Tr((\mathbf{B}(t))^m).$$

It suffices to prove that $f_m(t)$ is a logconvex function of t . Indeed, then $\frac{1}{m} \log f_m(t)$ is convex, hence the limit, $\log \lambda_{max}(t)$ is convex [25]. Here we use also the fact, that $\lambda_{max} \geq n$ [26], hence $f_m(t)$ can be bounded away from 0.

It remains to verify that $f(t) = f_m(t)$ is logconvex. From the definition of the parametrization (8) we see that

$$f(t) = \sum_{i=1}^N e^{c_i t + d_i},$$

where N is a positive integer and $c_i, d_i \in \mathbb{R}$ ($i, j = 1, 2, \dots, N$), all depend also on m . We have to prove that $\log f(t)$ is convex.

Lemma 1. The function $f(t)$ above is logconvex.

Proof of Lemma 1: The convexity of $\log f(t)$ is to be proved.

$$[\log f(t)]' = \frac{f'(t)}{f(t)}$$

$$[\log f(t)]'' = \frac{f''(t)f(t) - f'(t)f'(t)}{f^2(t)}$$

It is enough to prove the non-negativity of the numerator, that is, the inequality $f''(t)f(t) - f'(t)f'(t) \geq 0$.

Since

$$f'(t) = \sum_{i=1}^N c_i e^{c_i t + d_i},$$

$$f''(t) = \sum_{i=1}^N c_i^2 e^{c_i t + d_i},$$

we have

$$\begin{aligned} & \left(\sum_{i=1}^N c_i^2 e^{c_i t + d_i} \right) \left(\sum_{i=1}^N e^{c_i t + d_i} \right) - \left(\sum_{i=1}^N c_i e^{c_i t + d_i} \right)^2 = \\ & \sum_{i \neq j} (c_i^2 + c_j^2 - 2c_i c_j) e^{(c_i + c_j)t + d_i + d_j} \geq 0, \end{aligned}$$

which completes the proof of Lemma 1 as well as of Proposition 1. \square

Corollary 1. The tools of convex optimization can be used for solving the eigenvalue minimization problem (4).

Proposition 2: $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ is either strictly convex or constant along any line in the \mathbf{y} space.

Proof of Proposition 2: Consider the parametrization of a line in the \mathbf{y} space according to (8) and let

$$g(t) = \lambda_{max}(\mathbf{B}(t)), \quad t \in \mathbb{R}.$$

The function g is logconvex and we shall show that it is either strictly convex or constant.

Assume that g is not strictly convex. Then there exist $t_1, t_2 \in \mathbb{R}, t_1 < t_2$ and $0 < \kappa_0 < 1$ such that

$$g(\kappa_0 t_1 + (1 - \kappa_0)t_2) = \kappa_0 g(t_1) + (1 - \kappa_0)g(t_2). \quad (9)$$

Since the logarithmic function is strictly concave, we get from (9) that

$$\begin{aligned} \log g(\kappa_0 t_1 + (1 - \kappa_0)t_2) &= \log(\kappa_0 g(t_1) + (1 - \kappa_0)g(t_2)) \geq \\ &\geq \kappa_0 \log g(t_1) + (1 - \kappa_0) \log g(t_2), \end{aligned} \quad (10)$$

furthermore, the inequality in (10) holds as an equality if and only if $g(t_1) = g(t_2)$. On the other hand, from the logconvexity of g we have

$$\log g(\kappa_0 t_1 + (1 - \kappa_0)t_2) \leq \kappa_0 \log g(t_1) + (1 - \kappa_0) \log g(t_2), \quad (11)$$

Now, (10) and (11) imply that $g(t_1) = g(t_2)$ and taking (9) also into consideration:

$$g(\kappa_0 t_1 + (1 - \kappa_0)t_2) = g(t_1) = g(t_2).$$

Since g is convex, $g(t)$ is constant on the interval $[t_1, t_2]$. Let

$$\Lambda = g(t_1).$$

Then Λ is the minimal value of g over the real line \mathbb{R} . Let

$$S = \{t \in \mathbb{R} | g(t) = \Lambda\}. \quad (12)$$

Clearly $[t_1, t_2] \subseteq S$, S is a convex and close subset of \mathbb{R} , furthermore, since $t_1 < t_2$, S has a nonempty interior.

If $S = \mathbb{R}$, we are done. Otherwise, we have either $\max\{t | t \in S\} < \infty$ or $\min\{t | t \in S\} > -\infty$. It suffices to detail the proof for one of these cases. Assume that

$$\bar{t} = \max\{t | t \in S\} < \infty.$$

Since $\Lambda = g(t) = \lambda_{max}(\mathbf{B}(t))$, $\forall t \in S$, we have

$$\det(\mathbf{B}(t) - \Lambda \mathbf{I}) = 0, \quad \forall t \in S. \quad (13)$$

From the parametrization (8), we obtain

$$\det(\mathbf{B}(t) - \Lambda \mathbf{I}) = \alpha_0 + \sum_{l=1}^L \alpha_l e^{\beta_l t}, \quad (14)$$

where the values of L and α_l, β_l come from the expansion of the determinant, and they depend on $B(t)$ and Λ . Let

$$p(t) = \alpha_0 + \sum_{l=1}^L \alpha_l e^{\beta_l t}.$$

$p(t)$ is an analytical function, which equals to zero on a segment $[t_1, t_2]$. By the Identity Theorem for complex analytic functions (Theorem 1.3.7 in Ash [1]) we have $p(t) = \det(\mathbf{B}(t) - \Lambda \mathbf{I}) = 0$ for all $t \in \mathbb{R}$. This means that Λ is an eigenvalue of $\mathbf{B}(t)$ for all $t \in \mathbb{R}$. However, due to the properties that Perron eigenvalue of a positive matrix has multiplicity 1 (see Theorem 37.3 in [24]) and that the eigenvalues change continuously with t even when multiplicities are taken into consideration (see for example Rouché's Theorem in [1]), it cannot happen that Λ is the Perron eigenvalue in $t = \bar{t}$, but it is not for any $t > \bar{t}$. Consequently, we cannot have $\bar{t} < \infty$, thus $S = \mathbb{R}$ and

$$g(t) = \lambda_{max}(\mathbf{B}(t)) = \Lambda$$

for all $t \in \mathbb{R}$. This completes the proof of Proposition 2. \square

The main result of this section is the following uniqueness theorem.

Theorem 2: The optimal solution of problem (4) is unique if and only if the graph G corresponding to the incomplete pairwise comparison matrix is connected.

Proof of Theorem 2: Necessity is based on elementary linear algebra. If G is not connected, then the incomplete pairwise comparison matrix \mathbf{A} can be rearranged by simultaneous changes of the rows and the columns into a decomposable form \mathbf{D} :

$$\mathbf{D} = \left(\begin{array}{c|c} \mathbf{J} & \mathbf{X} \\ \hline \mathbf{X}' & \mathbf{K} \end{array} \right), \quad (15)$$

where the square matrices \mathbf{J}, \mathbf{K} contain all the known elements of the incomplete pairwise comparison matrix \mathbf{A} and \mathbf{X} contains variables only. Matrix \mathbf{X}' contains the componentwise reciprocals of the variables in \mathbf{X} . Note that matrices \mathbf{J}, \mathbf{K} may also contain variables.

Assume that \mathbf{J} is a $u \times u$ matrix. Let $\alpha > 0$ be an arbitrary scalar and $\mathbf{P} = \text{diag}(\underbrace{\alpha, \alpha, \dots, \alpha}_u, \underbrace{1, 1, \dots, 1}_{n-u})$. Then, the similarity transformation by \mathbf{P} results in the matrix

$$\mathbf{PDP}^{-1} = \mathbf{E} = \left(\begin{array}{c|c} \mathbf{J} & \alpha \cdot (\mathbf{X}) \\ \hline \frac{1}{\alpha} \cdot (\mathbf{X}') & \mathbf{K} \end{array} \right),$$

For any fixed values of the variables in \mathbf{X} the matrices \mathbf{A}, \mathbf{D} and \mathbf{E} are similar. Since $\alpha > 0$ is arbitrary, one may construct an infinite number of completions of \mathbf{A} having the same eigenvalues, including the largest one.

For sufficiency, the directed graph representation introduced in Section 2 is considered.

We call a sequence of integers $i_0, i_1, \dots, i_{k-1}, i_0$ a *closed walk of length k* , provided that $0 < i_j \leq n$ holds for $j = 0, \dots, k-1$.

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix and $\gamma = i_0, i_1, \dots, i_{k-1}, i_0$ be a closed walk. The *value* v_γ of γ is defined as the product of the entries of \mathbf{A} along

the walk:

$$v_\gamma := a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} i_0}.$$

Lemma 2. Let $\mathbf{A} = [a_{ij}]$ be a strictly positive $n \times n$ matrix and $\gamma = i_0, i_1, \dots, i_{k-1}, i_0$ be a closed walk of length k . Then $\lambda_{max}(\mathbf{A}) \geq (v_\gamma)^{\frac{1}{k}}$.

Proof of Lemma 2: The formula

$$\lambda_{max}(\mathbf{A}) = \lim_{m \rightarrow \infty} \sqrt[m]{Tr(\mathbf{A}^m)}$$

is used again.

Set $m = \ell k$, where ℓ is a positive integer. Then $Tr(\mathbf{A}^m) = \sum_{\delta} v_{\delta}$, where the summation is for all the n^m closed walks δ of length m . The terms of the sum are positive by assumption. Let δ^* be the walk obtained by passing through γ exactly ℓ times. Now δ^* is a closed walk of length m and $v_{\delta^*} = v_\gamma^\ell$. We infer that

$$\sqrt[m]{Tr(\mathbf{A}^m)} > \sqrt[m]{v_{\delta^*}} = \sqrt[\ell k]{v_\gamma^\ell} = \sqrt[k]{v_\gamma}.$$

The claim follows by taking $\ell \rightarrow \infty$. Lemma 2 is proved. \square

Lemma 3. Let the graph G of the incomplete pairwise comparison matrix \mathbf{A} be connected. Let x_h , ($1 \leq h \leq d$) be one of the missing elements and (l, m) denote the position of x_h in the matrix. Then

$$\lambda_{max}(\mathbf{A}(\mathbf{x})) \geq \max \left\{ (K x_h)^{\frac{1}{k}}, \left(\frac{1}{K x_h} \right)^{\frac{1}{k}} \right\},$$

where $k > 1$ is an integer, $K > 0$, and the values of k and K depend only on the known elements of the matrix and on l, m , but not on the value of x_h .

Proof of Lemma 3: Since G is connected, for some k there exists a path $l = i_0, i_1, \dots, i_{k-1} = m$ in G connecting l to m . Please note that the entries $a_{i_r i_{r+1}}$ are all specified values in \mathbf{A} for $r = 0, \dots, k-1$. Let γ be the following closed walk of length k

$$\gamma := i_0, i_1, \dots, i_{k-1}, i_0.$$

By Lemma 2 we have

$$\lambda_{max}(\mathbf{A}(\mathbf{x})) \geq (v_\gamma)^{\frac{1}{k}} = (K \cdot x_h)^{\frac{1}{k}},$$

where $K = a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{k-2} i_{k-1}}$ is a positive constant independent of the specializations of the variables x_i , ($i = 1, 2, \dots, d$). We obtain

$$\lambda_{max}(\mathbf{A}(\mathbf{x})) \geq \left(\frac{1}{K x_h} \right)^{\frac{1}{k}}$$

in a similar way, simply taking the reverse direction of the closed walk γ . Lemma 3 is proved. \square

Now we turn to the proof of Theorem 2. We are interested in $\Lambda := \min \lambda_{max}(\mathbf{A})$, where the minimum is taken over all (numerical) reciprocal matrices which are specializations of \mathbf{A} . To put it simply, we write positive real numbers b_{ij} in the place of the variables x_h ($h = 1, 2, \dots, d$) in all possible ways while maintaining the relations $b_{ij}b_{ji} = 1$, and take the minimum of the Perron roots of the reciprocal matrices obtained.

As before, we introduce the parametrization

$$x_1 = e^{y_1}, \quad x_2 = e^{y_2}, \quad \dots \quad x_d = e^{y_d}. \quad (16)$$

This way $\mathbf{A}(\mathbf{x}) = \mathbf{B}(\mathbf{y})$ is parameterized by vectors $\mathbf{y} = (y_1, \dots, y_d)$ from the space \mathbb{R}^d . Set

$$\Lambda := \min\{\lambda_{max}(\mathbf{B}(\mathbf{y})) : \mathbf{y} \in \mathbb{R}^d\}, \quad (17)$$

and

$$S = \{\mathbf{y} \in \mathbb{R}^d : \lambda_{max}(\mathbf{B}(\mathbf{y})) = \Lambda\}. \quad (18)$$

It follows from the connectedness of the graph G that the minimum is indeed attained in the definition of Λ . Assume for contradiction, that infimum exists only, then there is a sequence of vectors in \mathbb{R}^d , denoted by $\{\mathbf{y}^{(i)}\}_{i=1, \dots, \infty}$, such that

$$\lim_{i \rightarrow \infty} \lambda_{max}(\mathbf{B}(\mathbf{y}^{(i)})) = \Lambda$$

and

$$\sup\{y_h^{(i)}\}_{i=1, \dots, \infty} = \infty, \quad (19)$$

or

$$\inf\{y_h^{(i)}\}_{i=1, \dots, \infty} = -\infty. \quad (20)$$

for some h ($1 \leq h \leq d$). It follows from reciprocal property of pairwise comparison matrices that (19) and (20) are practically equivalent, therefore, it is sufficient to discuss (19). After possibly renaming we may assume that

$$\lim_{i \rightarrow \infty} y_h^{(i)} = \infty,$$

which contradicts Lemma 3 by choosing $x_h^{(i)} = e^{y_h^{(i)}}$, since Λ is a fixed finite number.

We shall prove that if $|S| > 1$, then the graph G of \mathbf{A} is not connected. To this end, assume first that $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$ are two different points of S . Let L

be the line passing through \mathbf{p} and \mathbf{q} . It follows from Proposition 2 that $L \subseteq S$.

From $L \subseteq S$ we obtain that there exists an undefined position (l, m) in the upper triangular part of matrix \mathbf{A} , such that for every real $M > 0$ there exists a completion $\mathbf{C} = [c_{ij}]$ of \mathbf{A} (a complete pairwise comparison matrix, which agrees with \mathbf{A} everywhere, where a_{ij} has been specified), such that $\lambda_{max}(\mathbf{C}) = \Lambda$ and $c_{lm} > M$. It contradicts to Lemma 3 by setting $x_h = c_{lm}$, because then $\lambda_{max}(\mathbf{C})$ will be unbounded if c_{lm} is arbitrarily large. This completes the proof of Theorem 2. \square

Corollary 3. If the graph G corresponding to the incomplete pairwise comparison matrix is connected, then by using the parametrization (16), problem (4) is transformed into a strictly convex optimization problem. Moreover, taking $\lambda_{max}(\mathbf{A}(\mathbf{x}))$ at an arbitrary $\mathbf{x} > \mathbf{0}$ as an upper bound of the optimal value of (4), a positive lower and an upper bound can be immediately obtained for x_h , ($1 \leq h \leq d$) from Lemma 3. This makes it possible to search for the optimal solution of (4) (and (17)) over a d -dimensional rectangle. We propose an algorithm for solving (4) in Section 5.

Consider the graph G corresponding to the incomplete pairwise comparison matrix \mathbf{A} . A subgraph G' of G is called a *connected component* of G if it is a maximal connected subgraph of G . Clearly, G can be divided into a finite number of disjoint connected components. Let G_1, \dots, G_s denote the connected components of G . The graph G is connected when $s = 1$. In this case, according to Theorem 2, the optimal solution of problem (4) (thus of (17)) is unique. The next theorem extends this result to the general case, namely, proves that the minimum of (4) (equivalently Λ of (17)) exists and characterizes the set of the optimal solutions, i.e. S of (18) in the \mathbf{y} space.

Theorem 3: The function $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ attains its minimum over \mathbb{R}^d , and the optimal solutions constitute an $(s-1)$ -dimensional affine set of \mathbb{R}^d , where s is the number of the connected components in G .

Proof of Theorem 3: If $s = 1$, we are done. By Theorem 2, there is a single optimal solution, and itself is a 0-dimensional affine set.

We turn now to the case $s > 1$. First, we show that $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ attains its minimum over \mathbb{R}^d . An idea from the proof of Theorem 2 is used here again. For $l = 1, \dots, s-1$, applying necessary simultaneous changes of the rows and the columns, the incomplete pairwise comparison matrix \mathbf{A} can be rearranged into the decomposable form \mathbf{D} of (15) such that the rows and

the columns of \mathbf{J} are associated with the nodes of the connected components G_1, \dots, G_l , the rows and the columns of \mathbf{K} are associated with the nodes of the connected components G_{l+1}, \dots, G_s , and \mathbf{X} consists of the variables whose rows and columns are associated with the nodes of G_1, \dots, G_l and G_{l+1}, \dots, G_s , respectively. Here again, \mathbf{X}' consists of the reciprocals of the variables in \mathbf{X} .

As shown in the proof of Theorem 2, for any $\mathbf{x} \in \mathbb{R}_+^d$, the value of $\lambda_{max}(\mathbf{A}(\mathbf{x}))$ does not change if all the entries of \mathbf{x} belonging to the block \mathbf{X} in (15) are multiplied by the same $\alpha > 0$. Choose a variable x_{i_l} from the block \mathbf{X} arbitrarily such that its row and column are associated with G_l and G_{l+1} , respectively. We can fix the value of x_{i_l} at any positive level, say let $x_{i_l} = v_{i_l}$. Then, for any $\mathbf{x} \in \mathbb{R}_+^d$ the vector $\bar{\mathbf{x}} \in \mathbb{R}_+^d$ obtained from \mathbf{x} by multiplying all the entries from the block \mathbf{X} by $\alpha = v_{i_l}/x_{i_l}$ has the properties $\bar{x}_{i_l} = v_{i_l}$ and $\lambda_{max}(\mathbf{A}(\bar{\mathbf{x}})) = \lambda_{max}(\mathbf{A}(\mathbf{x}))$.

Repeat the above idea for $l = 1, \dots, s-1$. In this way, we can choose variables x_{i_l} , $l = 1, \dots, s-1$, such that they belong to missing elements of \mathbf{A} , the row and column of x_{i_l} are associated with G_l and G_{l+1} , respectively; furthermore, for any $\mathbf{x} \in \mathbb{R}_+^d$ and for fixed values v_{i_l} , $l = 1, \dots, s-1$, a vector $\bar{\mathbf{x}} \in \mathbb{R}_+^d$ can be easily constructed such that $\bar{x}_{i_l} = v_{i_l}$, $l = 1, \dots, s-1$, and $\lambda_{max}(\mathbf{A}(\bar{\mathbf{x}})) = \lambda_{max}(\mathbf{A}(\mathbf{x}))$.

Thus, if we are interested only in finding an optimal solution of (4), it suffices to write arbitrary positive values into those missing entries of matrix \mathbf{A} which belong to $x_{i_1}, \dots, x_{i_{s-1}}$, and of course, the reciprocal values into the appropriate entries in the lower triangular part of \mathbf{A} . Let $\tilde{\mathbf{A}}$ denote the pairwise comparison matrix obtained in this way. Matrix $\tilde{\mathbf{A}}$ has $d - s + 1$ missing entries in the upper triangular part, and the graph G associated with $\tilde{\mathbf{A}}$ is connected. Thus, the problem

$$\min_{\tilde{\mathbf{x}} > \mathbf{0}} \lambda_{max}(\tilde{\mathbf{A}}(\tilde{\mathbf{x}})) \quad (21)$$

has a unique optimal solution, where $\tilde{\mathbf{x}}$ is the $(d - s + 1)$ -dimensional vector associated with the missing entries in the upper triangular part of $\tilde{\mathbf{A}}$. It is easy to see that by completing the optimal solution of (21) by the entries $x_{i_1}, \dots, x_{i_{s-1}}$ fixed in \mathbf{A} , we obtain an optimal solution of (4), moreover, due to (16), an optimal solution where $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ attains its minimum over \mathbb{R}^d , i.e. Λ of (17) exists and S of (18) is not empty.

We know that $|S| > 1$ if and only if $s > 1$. Let $\bar{\mathbf{y}}, \hat{\mathbf{y}} \in S$, $\bar{\mathbf{y}} \neq \hat{\mathbf{y}}$. Since $\lambda_{max}(\mathbf{B}(\bar{\mathbf{y}})) = \lambda_{max}(\mathbf{B}(\hat{\mathbf{y}})) = \Lambda$, we have $\lambda_{max}(\mathbf{B}(\mathbf{y})) = \Lambda$ along the entire line passing through $\bar{\mathbf{y}}$ and $\hat{\mathbf{y}}$. This comes from Proposition 2 since if $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ was strictly convex along the line, we would have $\lambda_{max}(\mathbf{B}(\mathbf{y})) < \Lambda$ for any interior point of the segment $[\bar{\mathbf{y}}, \hat{\mathbf{y}}]$ contradicting the minimality of

Λ. Consequently, for any $\bar{\mathbf{y}}, \hat{\mathbf{y}} \in S$, $\bar{\mathbf{y}} \neq \hat{\mathbf{y}}$, the line passing through $\bar{\mathbf{y}}$ and $\hat{\mathbf{y}}$ lies in S , thus S is an affine set in the \mathbf{y} space.

The affine set S can be written in the form of the solution set of a finite system of linear equalities

$$\mathbf{F}\mathbf{y} = \mathbf{f}, \quad (22)$$

see [25]. For the sake of simplicity, assume that $i_l = l$, $l = 1, \dots, s - 1$, and for any $\mathbf{y} \in \mathbb{R}^d$, let $\mathbf{y}^{(1)}$ be the vector of the first $s - 1$ elements, and let $\mathbf{y}^{(2)}$ be the vector of the last $d - s + 1$ elements of \mathbf{y} , i.e. $\mathbf{y} = (\mathbf{y}^{(1)T}, \mathbf{y}^{(2)T})^T$. We know that for any $\mathbf{y}^{(1)}$ there exists a unique $\mathbf{y}^{(2)}$ such that $\mathbf{y} = (\mathbf{y}^{(1)T}, \mathbf{y}^{(2)T})^T \in S$. Let $\mathbf{y}^{(2)} = h(\mathbf{y}^{(1)})$ denote this relation. Since for any $\bar{\mathbf{y}}^{(1)}, \hat{\mathbf{y}}^{(1)} \in \mathbb{R}^{s-1}$ and $\alpha \in \mathbb{R}$, the vectors $(\bar{\mathbf{y}}^{(1)T}, h(\bar{\mathbf{y}}^{(1)})^T)^T$, $(\hat{\mathbf{y}}^{(1)T}, h(\hat{\mathbf{y}}^{(1)})^T)^T$ and $((\alpha\bar{\mathbf{y}}^{(1)} + (1 - \alpha)\hat{\mathbf{y}}^{(1)})^T, (\alpha h(\bar{\mathbf{y}}^{(1)}) + (1 - \alpha)h(\hat{\mathbf{y}}^{(1)}))^T)^T$ are solutions of (22), we obtain that

$$h(\alpha\bar{\mathbf{y}}^{(1)} + (1 - \alpha)\hat{\mathbf{y}}^{(1)}) = \alpha h(\bar{\mathbf{y}}^{(1)}) + (1 - \alpha)h(\hat{\mathbf{y}}^{(1)}),$$

i.e. h is a linear function. Let $\mathbf{e}^{(l)}$ denote the l -th unit vector. It is easy to see that the s vectors $(\mathbf{0}^T, h(\mathbf{0})^T)^T$, $(\mathbf{e}^{(l)T}, h(\mathbf{e}^{(l)})^T)^T$, $l = 1, \dots, s - 1$, are affinely independent, and their affine hull is the solution set of (22), i.e. S . Consequently, S is a $(s - 1)$ -dimensional affine set, see [25]. This completes the proof of Theorem 3. \square

Corollary 4. It follows from Theorem 3 that if we are interested only in finding an optimal solution of (4), it suffices to solve (21) that can be, as shown before, reduced to the minimization of a strictly convex function over a rectangle. If we are interested in generating the whole set S , the proof of Theorem 3 shows how to construct affinely independent vectors whose affine hull is S .

Although $\lambda_{max}(\mathbf{A}(\mathbf{x}))$ is non-convex over \mathbb{R}_+^d it is either constant or unimodal over any line parallel to an axis, i.e., when a single entry of the pairwise comparison matrix varies. This comes directly from Proposition 2 and the parametrization (16). The advantage of this property will be used in Section 5 when the method of cyclic coordinates will be applied for solving (4).

Remark 2. The Perron eigenvalue of a pairwise comparison matrix is non-convex function of its elements. Let \mathbf{Q} be a 3×3 pairwise comparison matrix of variable x as follows:

$$\mathbf{Q} = \begin{pmatrix} 1 & 2 & x \\ 1/2 & 1 & 4 \\ 1/x & 1/4 & 1 \end{pmatrix}.$$

$\lambda_{max}(\mathbf{Q}(x))$ is plotted in Figure 3.a. However, by using the exponential scaling $x = e^t$, $\lambda_{max}(\mathbf{Q}(e^t))$ becomes a convex function of t (Figure 3.b).

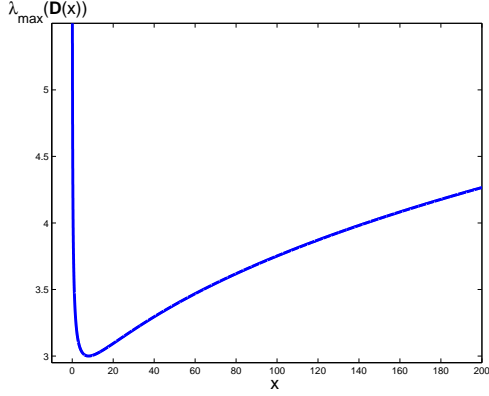


Figure 3.a Non-convexity of the function $x \mapsto \lambda_{max}(\mathbf{Q}(x))$

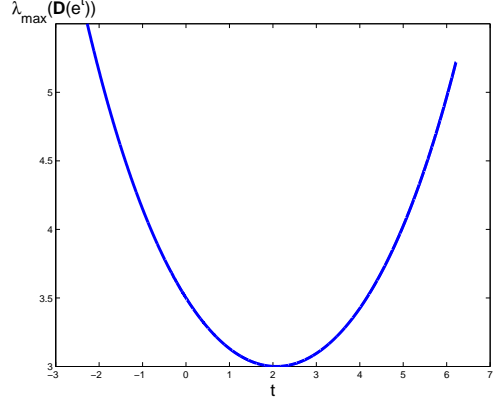


Figure 3.b Convexity of the function $t \mapsto \lambda_{max}(\mathbf{Q}(e^t))$

4 The Logarithmic Least Squares Method for incomplete matrices

In this section, the extension of the *LLSM* problem (3) for incomplete matrices is discussed. Having an incomplete pairwise comparison matrix \mathbf{A} , one should consider the terms for those pairs (i, j) only for which a_{ij} is given:

$$\min \sum_{\substack{\epsilon(i,j) \in E \\ 1 \leq i < j \leq n}} \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2 + \left[\log a_{ji} - \log \left(\frac{w_j}{w_i} \right) \right]^2 \quad (23)$$

$$\sum_{i=1}^n w_i = 1, \quad (24)$$

$$w_i > 0, \quad i = 1, 2, \dots, n. \quad (25)$$

For the reader's convenience, each pair of terms related to a_{ij} and a_{ji} is written jointly in the objective function (23). By definition, the terms related to $i = j$ equal to 0, therefore, they are omitted from the objective function.

Theorem 4: The optimal solution of the incomplete *LLSM* problem (23)-(25) is unique if and only if graph G corresponding to the incomplete pairwise comparison matrix is connected.

Proof: Since the value of the objective function is the same in an arbitrary point \mathbf{w} and in $\alpha\mathbf{w}$ for any $\alpha > 0$ we can assume that

$$w_n = 1, \quad (26)$$

instead of normalization (24). Following Kwiesielewicz's computations for complete matrices [20](pp. 612-613), let us introduce the variables

$$r_{ij} = \log a_{ij} \quad i, j = 1, 2, \dots, n \text{ and } e(i, j) \in E; \quad (27)$$

$$y_i = \log w_i \quad i = 1, 2, \dots, n. \quad (28)$$

Note that $y_n = \log w_n = 0$. The new problem, equivalent to the original one, is as follows:

$$\min \sum_{\substack{i, j: \\ e(i, j) \in E \\ 1 \leq i < j < n}} (r_{ij} - y_i + y_j)^2 + \sum_{\substack{i: \\ e(i, n) \in E}} (r_{in} - y_i)^2 \quad (29)$$

This problem is unconstrained ($y_1, y_2, \dots, y_{n-1} \in \mathbb{R}$). The first-order conditions of optimality can be written as:

$$\begin{pmatrix} d_1 & -1 & 0 & \dots & -1 \\ -1 & d_2 & -1 & \dots & 0 \\ 0 & -1 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & d_{n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} -\sum r_{i1} \\ -\sum r_{i2} \\ -\sum r_{i3} \\ \vdots \\ -\sum r_{i,n-1} \end{pmatrix}, \quad (30)$$

where d_i denotes the degree of the i -th node in graph G ($0 < d_i \leq n - 1$), and the (i, j) position equals to -1 if $e(i, j) \in E$ and 0 if $e(i, j) \notin E$. Note that the summation $\sum r_{ik}$ denotes

$$\sum_{\substack{i: \\ e(i, k) \in E}} r_{ik}$$

in each component ($k = 1, 2, \dots, n - 1$) of the right-hand side.

The matrix of the coefficients of size $(n-1) \times (n-1)$ can be augmented by the n -th row and column based on the same rules as above: (n, n) position equals to d_n , the degree of n -th node in graph G ; (i, n) and (n, i) positions equal to -1 if $e(i, n) \in E$ (or, equivalently, $e(n, i) \in E$) and 0 if $e(i, n) \notin E$ (or, equivalently, $e(n, i) \notin E$) for all $i = 1, 2, \dots, n-1$. The augmented $n \times n$ matrix has the same rank as the $(n-1) \times (n-1)$ matrix of the coefficients in (30) because the number of -1 's in the i -th row/column is equal to d_i ($i = 1, 2, \dots, n$) and the n -th column/row is the negative sum of the first $n-1$ columns/rows.

Now, the augmented matrix of size $n \times n$, is exactly the Laplace-matrix of graph G , denoted by $\mathbf{L}_{n \times n}$. Some of the important properties of Laplace-matrix $\mathbf{L}_{n \times n}$ are as follows (see, e.g., Section 6.5.6 in [11]):

- (a) eigenvalues are real and non-negative;
- (b) the smallest eigenvalue, $\lambda_1 = 0$;
- (c) the second smallest eigenvalue, $\lambda_2 > 0$ if and only if the graph is connected; λ_2 is called Laplace-eigenvalue of the graph;
- (d) rank of $\mathbf{L}_{n \times n}$ is $n-1$ if and only if graph G is connected.

Let $\mathbf{L}_{(n-1) \times (n-1)}$ denote the upper-left $(n-1) \times (n-1)$ submatrix of $\mathbf{L}_{n \times n}$. The solution of the system of first-order conditions uniquely exists if and only if matrix $\mathbf{L}_{(n-1) \times (n-1)}$ is of full rank, that is, graph G is connected.

Since the equations of the first-order conditions are linear, the Hessian matrix is again $\mathbf{L}_{(n-1) \times (n-1)}$, which is always symmetric, and, following from the properties of Laplace-matrix above, it is positive definite if and only if graph G is connected. \square

Remark 3. The uniqueness of the solution depends only on the positions of comparisons (the structure of the graph G), and does not depend on the values of comparisons.

Remark 4. In the case of an $n \times n$ complete pairwise comparison matrix, the solution of (30) is as follows:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{pmatrix}^{-1} \begin{pmatrix} -\sum r_{i1} \\ -\sum r_{i2} \\ -\sum r_{i3} \\ \vdots \\ -\sum r_{i,n-1} \end{pmatrix}, \quad (31)$$

where $\sum r_{ik}$ consists of $n - 1$ elements for every k ($1 \leq k \leq n - 1$) and equals to the logarithm of the product of the k -th row's elements of the complete pairwise comparison matrix. Applying (28) and (26), then renormalizing the weight vector by (24), the optimal solution of (23)-(25) is the well known geometric mean, also mentioned in the introduction (3).

Remark 5. Problem (29) has some similarities with additive models of pairwise comparisons, e.g., with the constraints in Xu's goal programming model [33](LOP1 on p. 264). However, Xu's objective function is linear, while (29) is quadratic. Fedrizzi and Giove considers several penalty functions originated from equations for consistency. One of their models [9]((4) on p. 304) is also similar, but not equivalent, to our (29).

Remark 6. Theorem 4 shows similarity to Theorem 1 of Fedrizzi and Giove [9](p. 310) regarding the uniqueness of the optimal solution and the connectedness of graph G . In both problems a sum of quadratic functions associated with the edges of G is minimized, although the functions are different in the two approaches.

Example 2. As an illustration to the incomplete *LLSM* problem and our proposed solution, we introduce the partial matrix \mathbf{M} below. In fact, \mathbf{M} is an incomplete modification of the frequently cited 'Buying a house' example by Saaty [27]. The undirected graph representation of the partial matrix \mathbf{M} is shown in Figure 4.

$$\mathbf{M} = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & * & 5 & * & 3 & * & 1/7 \\ 1/3 & * & 1 & * & 3 & * & 6 & * \\ 1/7 & 1/5 & * & 1 & * & 1/4 & * & 1/8 \\ 1/6 & * & 1/3 & * & 1 & * & 1/5 & * \\ 1/6 & 1/3 & * & 4 & * & 1 & * & 1/6 \\ 3 & * & 1/6 & * & 5 & * & 1 & * \\ 4 & 7 & * & 8 & * & 6 & * & 1 \end{pmatrix}. \quad (32)$$

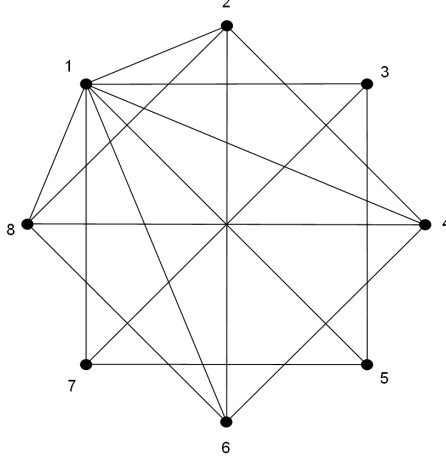


Figure 4. The undirected graph representation of the 8×8 incomplete pairwise comparison matrix \mathbf{M}

Using the notation of (27), the system of linear equations (30) is as follows:

$$\begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 4 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 3 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 4 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 3 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 & 4 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{pmatrix} = \begin{pmatrix} -\log(1/315) \\ -\log(7/3) \\ -\log(1/6) \\ -\log 1120 \\ -\log 90 \\ -\log 27 \\ -\log(2/5) \end{pmatrix}, \quad (33)$$

where the k -th ($k = 1, 2, \dots, 7$) component of the right-hand side is computed as the negative sum of the logarithms of the k -th column's elements in matrix \mathbf{M} in (32), which also equals to the logarithm of the product of the k -th row's elements.

The solution of (33) is $y_1 = -0.6485$, $y_2 = -1.6101$, $y_3 = -0.6485$, $y_4 = -2.8449$, $y_5 = -2.2214$, $y_6 = -2.0998$, $y_7 = -0.8674$.

Using (28), then turning back to the normalization (24), the optimal solution of the incomplete *LLSM* problem concerning matrix \mathbf{M} is as follows:
 $w_1^{LLSM} = 0.1770$, $w_2^{LLSM} = 0.0676$, $w_3^{LLSM} = 0.1770$, $w_4^{LLSM} = 0.0197$,
 $w_5^{LLSM} = 0.0367$, $w_6^{LLSM} = 0.0415$, $w_7^{LLSM} = 0.1422$, $w_8^{LLSM} = 0.3385$.

5 An algorithm for the λ_{max} -optimal completion

In this section, an algorithm is proposed for finding the best completion of an incomplete pairwise comparison matrix by the generalization of the eigenvector method. It is shown in Section 3 that the eigenvalue minimization leads to a convex optimization problem. However, the derivatives of λ_{max} as functions of missing elements are not easily and explicitly computable. Therefore, one should provide an algorithm without derivatives. In our approach, the iterative method of cyclic coordinates is proposed. Let d denote the number of missing elements. Write variables x_1, x_2, \dots, x_d in place of the missing elements. Let $x_i^{(0)}, i = 1, 2, \dots, d$ be arbitrary positive numbers, we will use them as initial values. Each iteration of the algorithm consists of d steps.

In the first step of the first iteration, let variable x_1 be free and the other variables be fixed to the initial values: $x_i = x_i^{(0)}, (i = 2, 3, \dots, d)$. The aim is to minimize λ_{max} as a univariate function of x_1 . Since the eigenvalue optimization can be transformed into a multivariate convex optimization problem, it remains convex when restricting to one variable. Let $x_1^{(1)}$ denote the optimal solution computed by a univariate minimization algorithm.

In the second step of the first iteration x_2 is free, the other variables are fixed as follows: $x_1 = x_1^{(1)}, x_i = x_i^{(0)}, (i = 3, 4, \dots, d)$. Now we minimize λ_{max} in x_2 . Let $x_2^{(1)}$ denote the optimal solution.

After analogous steps, the d -th step of the first iteration is to minimize λ_{max} in x_d , where all other variables are fixed by the rule $x_i = x_i^{(1)}, (i = 1, 2, \dots, d - 1)$. The optimal solution is denoted by $x_d^{(1)}$ and it completes the first iteration of the algorithm.

In the second iteration the initial values computed in the first iteration are used. The univariate minimization problems are analogously written and solved.

The stopping criteria can be modified or adjusted in different ways. In our tests accuracy is set for 4 digits. The algorithm stops in the end of the k -th iteration if k is the smallest integer for which $\max_{i=1,2,\dots,d} \|x_i^k - x_i^{k-1}\| < T$, where T denotes the tolerance level.

The global convergence of cyclic coordinates is stated and proved, e.g., in ([22], pages 253-254).

In our tests, $x_i^{(0)} = 1, (i = 1, 2, \dots, d)$ and $T = 10^{-4}$ were applied. We

used the function *fminbnd* in Matlab v.6.5 for solving univariable minimization problems.

For the reader's convenience, our algorithm is presented on the 8×8 incomplete pairwise comparison matrix \mathbf{M} from the previous section (32).

The aim is to find the completion of \mathbf{M} for which λ_{max} is minimal. Write the variables x_i , ($i = 1, 2, \dots, 12$; $x_i \in \mathbb{R}_+$) in place of the missing elements and let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$.

$$\mathbf{M}(\mathbf{x}) = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & x_1 & 5 & x_2 & 3 & x_3 & 1/7 \\ 1/3 & 1/x_1 & 1 & x_4 & 3 & x_5 & 6 & x_6 \\ 1/7 & 1/5 & 1/x_4 & 1 & x_7 & 1/4 & x_8 & 1/8 \\ 1/6 & 1/x_2 & 1/3 & 1/x_7 & 1 & x_9 & 1/5 & x_{10} \\ 1/6 & 1/3 & 1/x_5 & 4 & 1/x_9 & 1 & x_{11} & 1/6 \\ 3 & 1/x_3 & 1/6 & 1/x_8 & 5 & 1/x_{11} & 1 & x_{12} \\ 4 & 7 & 1/x_6 & 8 & 1/x_{10} & 6 & 1/x_{12} & 1 \end{pmatrix}$$

The algorithm for the optimization problem

$$\min_{\mathbf{x}} \lambda_{max}(\mathbf{M}(\mathbf{x})) \quad (34)$$

is as follows. Let $x_i^{(k)}$ denote the value of x_i in the k -th step of the iteration, which has 12 substeps for each k .

For $k = 0$:

Let the initial points be equal to 1 for every variable:

$$x_i^{(0)} := 1 \quad (i = 1, 2, \dots, 12).$$

while $\max_{i=1,2,\dots,12} \|x_i^k - x_i^{k-1}\| > T$

$$x_i^{(k)} := \arg \min_{x_i} \lambda_{max}(\mathbf{M}(x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k-1)}, \dots, x_{12}^{(k-1)})), \quad i = 1, 2, \dots, 12$$

next k

end while

Table 1 presents the results of each substep of the first 20 iterations of the algorithm. Results reach the accuracy up to 4 digits in the 19-th iteration. After plotting the objective function's value during the iteration steps, it may be observed, that a significant decrease happens in the first iteration (Figure 5).

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	$x_5^{(k)}$	$x_6^{(k)}$	$x_7^{(k)}$	$x_8^{(k)}$	$x_9^{(k)}$	$x_{10}^{(k)}$	$x_{11}^{(k)}$	$x_{12}^{(k)}$
0	1	1	1	1	1	1	1	1	1	1	1	1
1	0.5761	1.5660	0.7158	4.7610	3.3936	0.7884	0.5895	0.1789	1.0599	0.1373	0.3029	0.3939
2	0.3889	1.8490	0.5119	9.1387	4.9089	0.5643	0.5321	0.1476	0.9444	0.1073	0.2966	0.3930
3	0.3427	1.7646	0.4763	9.7847	4.8026	0.5607	0.5324	0.1443	0.9184	0.1077	0.2947	0.3985
4	0.3343	1.7411	0.4706	9.8270	4.8078	0.5648	0.5307	0.1435	0.9225	0.1084	0.2932	0.4007
5	0.3323	1.7319	0.4688	9.8608	4.8260	0.5669	0.5285	0.1431	0.9262	0.1088	0.2924	0.4017
6	0.3314	1.7268	0.4678	9.8855	4.8373	0.5680	0.5271	0.1428	0.9283	0.1090	0.2919	0.4023
7	0.3308	1.7238	0.4672	9.9003	4.8437	0.5687	0.5263	0.1427	0.9295	0.1091	0.2916	0.4026
8	0.3305	1.7221	0.4668	9.9088	4.8474	0.5691	0.5259	0.1426	0.9302	0.1092	0.2914	0.4028
9	0.3303	1.7211	0.4666	9.9138	4.8496	0.5693	0.5256	0.1425	0.9307	0.1093	0.2913	0.4029
10	0.3302	1.7205	0.4665	9.9166	4.8508	0.5694	0.5255	0.1425	0.9309	0.1093	0.2913	0.4030
11	0.3301	1.7202	0.4664	9.9183	4.8515	0.5695	0.5254	0.1425	0.9310	0.1093	0.2913	0.4030
12	0.3301	1.7200	0.4664	9.9192	4.8520	0.5695	0.5253	0.1425	0.9311	0.1093	0.2912	0.4031
13	0.3300	1.7199	0.4664	9.9198	4.8522	0.5696	0.5253	0.1424	0.9311	0.1093	0.2912	0.4031
14	0.3300	1.7198	0.4664	9.9201	4.8523	0.5696	0.5253	0.1424	0.9312	0.1093	0.2912	0.4031
15	0.3300	1.7198	0.4664	9.9203	4.8524	0.5696	0.5253	0.1424	0.9312	0.1093	0.2912	0.4031
16	0.3300	1.7198	0.4664	9.9204	4.8525	0.5696	0.5253	0.1424	0.9312	0.1093	0.2912	0.4031
17	0.3300	1.7198	0.4664	9.9204	4.8525	0.5696	0.5253	0.1424	0.9312	0.1093	0.2912	0.4031
18	0.3300	1.7198	0.4663	9.9205	4.8525	0.5696	0.5253	0.1424	0.9312	0.1093	0.2912	0.4031
19	0.3300	1.7197	0.4663	9.9205	4.8525	0.5696	0.5253	0.1424	0.9312	0.1093	0.2912	0.4031
20	0.3300	1.7197	0.4663	9.9205	4.8525	0.5696	0.5253	0.1424	0.9312	0.1093	0.2912	0.4031

Table 1. The first 20 iterations of the algorithm applied to the 8×8 incomplete pairwise comparison matrix \mathbf{M}

Based on the results of the algorithm above, the optimal solution of (34), i.e., the λ_{max} -optimal completion of the 8×8 incomplete pairwise comparison matrix \mathbf{M} is as follows (completing numbers in place of the former missing elements are displayed up to 2 digits):

$$\mathbf{M}(\mathbf{x}^*) = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & \mathbf{0.33} & 5 & \mathbf{1.72} & 3 & \mathbf{0.47} & 1/7 \\ 1/3 & \mathbf{3.03} & 1 & \mathbf{9.92} & 3 & \mathbf{4.85} & 6 & \mathbf{0.57} \\ 1/7 & 1/5 & \mathbf{0.10} & 1 & \mathbf{0.53} & 1/4 & \mathbf{0.14} & 1/8 \\ 1/6 & \mathbf{0.58} & 1/3 & \mathbf{1.90} & 1 & \mathbf{0.93} & 1/5 & \mathbf{0.11} \\ 1/6 & 1/3 & \mathbf{0.21} & 4 & \mathbf{1.07} & 1 & \mathbf{0.29} & 1/6 \\ 3 & \mathbf{2.14} & 1/6 & \mathbf{7.02} & 5 & \mathbf{3.43} & 1 & \mathbf{0.40} \\ 4 & 7 & \mathbf{1.76} & 8 & \mathbf{9.15} & 6 & \mathbf{2.48} & 1 \end{pmatrix}$$

The normalized right eigenvector corresponding to the largest eigenvalue $\lambda_{max}(\mathbf{M}(\mathbf{x}^*)) = 9.2981$ is: $w_1^{EM} = 0.1894$, $w_2^{EM} = 0.0567$, $w_3^{EM} = 0.2116$, $w_4^{EM} = 0.0175$, $w_5^{EM} = 0.0319$, $w_6^{EM} = 0.0354$, $w_7^{EM} = 0.1509$, $w_8^{EM} = 0.3066$.

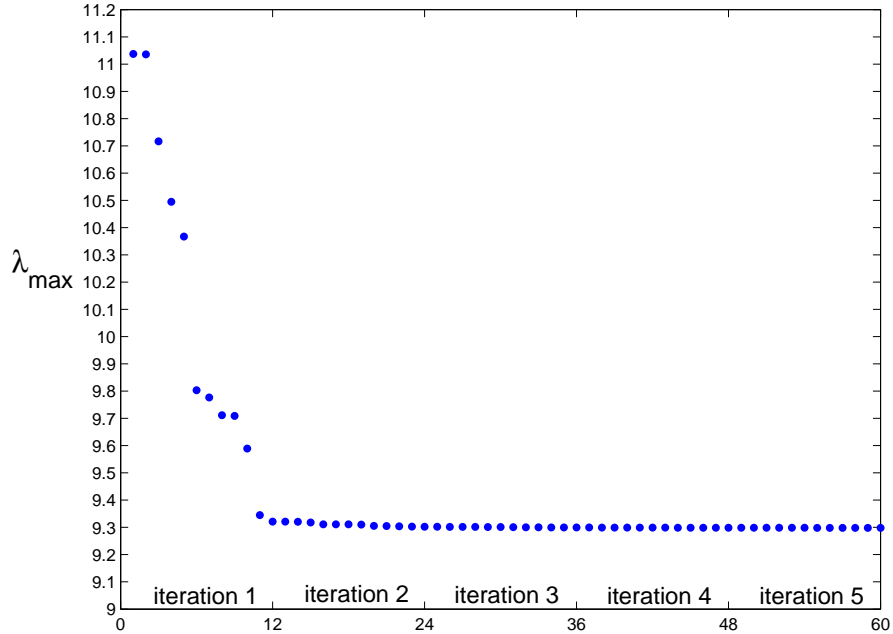


Figure 5. The decrease of $\lambda_{\max}(\mathbf{M})$ during the first 5 iterations

6 van Uden’s approximation as a starting point of the algorithm

An approximation of missing elements based on 3×3 submatrices, which themselves are also pairwise comparison matrices, also known as triads, was proposed by van Uden [31]. Suppose that b_{ij} is a missing element but a_{ik} and a_{jk} are given. In terms of the graph representation in Section 2 we have a triangle with one edge missing. This configuration (i, j, k) is a triad. It is natural to specify a value for b_{ij} by transitivity; set $a_{ij} = a_{ik}a_{kj}$.

If a missing element b_{ij} in the matrix (or, equivalently, a missing edge in the graph) can be approximated via several triads, the geometric mean of the approximations is applied.

Example 3. Let \mathbf{U} be a 4×4 incomplete pairwise comparison matrix with one missing element:

$$\mathbf{U} = \begin{pmatrix} 1 & 1 & 5 & 2 \\ 1 & 1 & 3 & 4 \\ 1/5 & 1/3 & 1 & * \\ 1/2 & 1/4 & * & 1 \end{pmatrix}$$

The missing element, denoted by x , is contained in two triads in which the other two elements are known:

$$\begin{pmatrix} 1 & 5 & 2 \\ 1/5 & 1 & x \\ 1/2 & 1/x & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 4 \\ 1/3 & 1 & x \\ 1/4 & 1/x & 1 \end{pmatrix}$$

The approximation of x using van Uden's rule is as follows:

$$\tilde{x} = \sqrt{\frac{2}{5} \cdot \frac{4}{3}} = 0.73029674334.$$

The optimal solution of the problem $\min_x \lambda_{max}(\mathbf{U}(x))$ resulted in by the algorithm in Section 5:

$$x^* = \mathbf{0.7302965066047},$$

which equals to van Uden's approximation up to 6 digits. Based on our numerical experience, if the number of missing elements is significantly smaller than the number of known elements, van Uden's rule ([31],[21]) provides a very good approximation for the missing elements, and gives suitable starting points for the λ_{max} -optimization algorithm as well. The mathematical justification of this important observation will be a topic of further research but is beyond the scope of this paper.

However, when the numbers of missing and known elements are of the same order, both starting points (1-s and van Uden's approximation) provide more or less the same rate of convergence. In the case of the incomplete 8×8 matrix \mathbf{M} , 22 iterations are needed in order to get the same accuracy starting from van Uden's initial point, while 19 iterations are enough when starting from 1-s.

7 Conclusion

A natural necessary and sufficient condition, the connectedness of the associated graph, is given for the uniqueness of the best completion of an incomplete pairwise comparison matrix regarding the Eigenvector Method and the Logarithmic Least Squares Method.

The eigenvalue optimization problem of the Eigenvector Method can be transformed into a convex, and, in the case of connected graph, into a strictly

convex optimization problem. Based on our algorithm proposed in Section 5, weights and CR -inconsistency can be computed from partial information. Moreover, the decision maker gets non-decreasing lower bound for the CR -inconsistency level in each step of the process of filling in the pairwise comparison matrix. This, especially in the case of a sharp jump, can be used for detecting misprints in real time.

In the Logarithmic Least Squares problem for incomplete matrices, the geometric means of the rows' elements play important role in the explicit computation of the optimal solution, like in the complete case.

The number of *necessary* pairwise comparisons (if it is smaller than $\frac{n(n-1)}{2}$ at all) depends on the characteristics of the real decision problem and provides an exciting topic of future research.

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