Bozóki, S., Fülöp, J., Rónyai, L. [2009]: Incomplete Pairwise Comparison Matrices in Multi-Attribute Decision Making,
In: Proceedings of the IEEE International Conference on Industrial Engineering and Engineering Management (IEEM),
Hong Kong, 8-11 Dec 2009, pp. 2256-2260., ISBN 978-1-4244-4870-8., DOI: 10.1109/IEEM.2009.5373064
http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=5373064&isnumber=5372873

Incomplete Pairwise Comparison Matrices in Multi-Attribute Decision Making

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Abstract – **An extension of the pairwise comparison matrix is considered when some comparisons are missing. A generalization of the eigenvector method for the incomplete case is introduced and discussed as well as the Logarithmic Least Squares Method. The uniqueness problem regarding both weighting methods is studied through the graph representation of pairwise comparison matrices. It is shown that the optimal completion/solution is unique if and only if the graph associated with the incomplete pairwise comparison matrix is connected. An algorithm is proposed for solving the eigenvalue minimization problem related to the generalization of the eigenvector method in the incomplete case. Numerical examples are presented for illustration of the methods discussed in the paper.**

Keywords – **Multi-attribute decision making, incomplete pairwise comparison matrix, eigenvalue optimization**

I. INTRODUCTION

In multi-attribute decision models, weighting the criteria and evaluating the alternatives with respect to the criteria are of the most important steps. In the paper, the method of pairwise comparison matrices [11] is discussed and generalized for the incomplete case. For simplicity, assume that the objects to compare are the importances of n criteria (the method is the same when comparing alternatives to each other or voting powers of the individuals in group decision making). A (complete) pairwise comparison matrix $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$ is a matrix of size $n\times n$ with the properties as follows:

$$a_{ij} > 0; \qquad (1)$$

$$a_{ii} = 1; \qquad (2)$$

$$a_{ij} = 1/a_{ji}, \qquad (3)$$

for $i,j=1,\ldots,n$, where a_{ij} is the numerical answer given by the decision maker for the question “How many times Criterion i is more important than Criterion j?” The weighting problem is to find the n-dimensional positive weight vector $\mathbf{w} = (w_1,w_2,\ldots,w_n)^T$ such that the appropriate ratios of the components of $\mathbf{w}$ reflect, or, at least, approximate all the a_{ij} values ($i,j=1,\ldots,n$), given by the decision maker. Eigenvector Method [11] and Logarithmic Least Squares Method [4,3] are two well-known and often cited weighting methods.

In the Analytic Hierarchy Process (AHP) [11] Eigenvector Method is applied and the approximation $\mathbf{w}^{EM}$ of $\mathbf{w}$ is defined by

$$\mathbf{A}\,\mathbf{w}^{EM} = \lambda_{\max}\,\mathbf{w}^{EM},$$

where $\lambda_{\max}$ denotes the maximal eigenvalue, also known as Perron eigenvalue, of $\mathbf{A}$ and $\mathbf{w}^{EM} = (w_1^{EM}, w_2^{EM},\ldots,w_n^{EM})^T$ denotes the right-hand side eigenvector of $\mathbf{A}$ corresponding to $\lambda_{\max}$. By Perron's theorem, $\mathbf{w}^{EM}$ is positive and unique up to a scalar multiplication [10]. The most often used normalization is $\Sigma w_i^{EM} = 1$.

A pairwise comparison matrix in is called *consistent* if the transitivity $a_{ij}\, a_{jk} = a_{ik}$ holds for all indices $i,j=1,\ldots,n$. Otherwise, the matrix is *inconsistent*. However, there are different levels of inconsistency, some of which are acceptable in solving real decision problems, some are not. Saaty [11] defined the inconsistency ratio CR as a positive linear transformation of $\lambda_{\max}$. It is well known that $CR = 0$ if and only if the matrix is consistent. According to Saaty, larger value of CR indicates higher level of inconsistency and the 10 percent rule ($CR \leq 0.10$) separates acceptable matrices from unacceptable ones.

The Logarithmic Least Squares Method (*LLSM*) gives $\mathbf{w}^{LLSM}$ as the optimal solution of:

$$\begin{aligned} &\min \Sigma_{i,j=1,\ldots,n} [\log(a_{ij}\, w_j / w_i)]^2 \\ &\Sigma_{i=1,\ldots,n}\, w_i = 1, \qquad\qquad (4) \\ &w_i > 0,\ \ i=1,\ldots,n. \end{aligned}$$

The optimization problem (4) is known to be solvable, and has a unique optimal solution, which can be explicitly computed by taking the geometric means of rows' elements [4,3].

Incomplete pairwise comparison matrix was defined by Harker [5]. It is of the same form as (1)–(3) but one ore more elements, denoted here by *, are not given:

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[1]Research was supported in part by OTKA grants K 60480, K 77420, NK 63066, NK 72845, K77476

$$\mathbf{A} = \begin{array}{|c|c|c|c|c|}\hline 1 & a_{12} & * & .. & a_{1n} \\ \hline 1/a_{12} & 1 & a_{23} & .. & * \\ \hline * & 1/\,a_{23} & 1 & .. & a_{3n} \\ \hline : & : & : & & : \\ \hline 1/\,a_{1n} & * & 1/\,a_{3n} & .. & 1 \\ \hline \end{array}$$

Variables $x_1,x_2,\ldots,x_d$ ($x_i > 0$, $i=1,\ldots,d$) are introduced for the missing elements in the upper triangular part of **A**. Their reciprocals, $1/x_1$, $1/x_2$, …, $1/x_d$ are written in the lower triangular part of **A**. Let **x** denote the vector (x_1, x_2 ,…, x_d). **A**(**x**) is a complete pairwise comparison matrix for any values of **x**. The total number of missing elements in matrix **A** is $2d$.

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}(x_1, x_2 ,\ldots, x_d) = \begin{array}{|c|c|c|c|c|}\hline 1 & a_{12} & x_1 & .. & a_{1n} \\ \hline 1/a_{12} & 1 & a_{23} & .. & x_d \\ \hline 1/x_1 & 1/a_{23} & 1 & .. & a_{3n} \\ \hline : & : & : & & : \\ \hline 1/a_{1n} & 1/x_d & 1/a_{3n} & .. & 1 \\ \hline \end{array}$$

The aim of the talk is to solve the incomplete versions of the Eigenvector Method and the Logarithmic Least Squares Method.

Based on the correspondence between the *CR* inconsistency and $\lambda_{\max}$ of a pairwise comparison matrix, the generalization of the Eigenvector Method for the incomplete case [2,12,13] is originated from the optimal solution of the eigenvalue minimization problem as follows:

$$\min \{ \lambda_{\max} (\mathbf{A}(\mathbf{x})) \mid \mathbf{x} > \mathbf{0} \}. \qquad (5)$$

The Logarithmic Least Squares Method for incomplete pairwise comparison matrices [7,14,8] is as follows:

$$\begin{aligned} &\min \Sigma_{e(i,j)\,\epsilon\, E} [\log(a_{ij}\, w_j / w_i)]^2 \\ &\Sigma_{i=1,\ldots,n}\, w_i = 1, \qquad (6) \\ &w_i > 0, \ i=1,\ldots,n. \end{aligned}$$

II. METHODOLOGY

Graph representation

Assume that the decision maker is asked to compare the relative importance of n criteria and s/he is filling in the pairwise comparison matrix. In each comparison, a direct relation is defined between two criteria, namely, the estimated ratio of their weights. However, two criteria, not compared yet, consequently, having no direct relation, can be in indirect relation, through further criteria and direct relations. It is a natural idea to associate graph structures to (in)complete pairwise comparison matrices.

Given an (in)complete pairwise comparison matrix **A** of size $n\times n$, graphs G is defined as follows: $G := (V,E)$, where $V =\{1, 2, \ldots, n\}$, the vertices correspond to the objects to compare and $E = \{ e(i,j) \mid a_{ij}$ and a_{ji} is given and $i\neq j \}$, the undirected edges correspond to the matrix elements. There are no edges corresponding to the missing elements in the matrix. G is an undirected graph.

Example. Let **C** be a 5×5 incomplete pairwise comparison matrix as follows:

$$\mathbf{C} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & * & * & 8 \\ \hline 1/2 & 1 & 3 & 6 & 7 \\ \hline * & 1/3 & 1 & 4 & * \\ \hline * & 1/6 & 1/4 & 1 & 5 \\ \hline 1/8 & 1/7 & * & 1/5 & 1 \\ \hline \end{array} \qquad (7)$$

Graph associated with matrix **C** is drawn in Fig. 1.

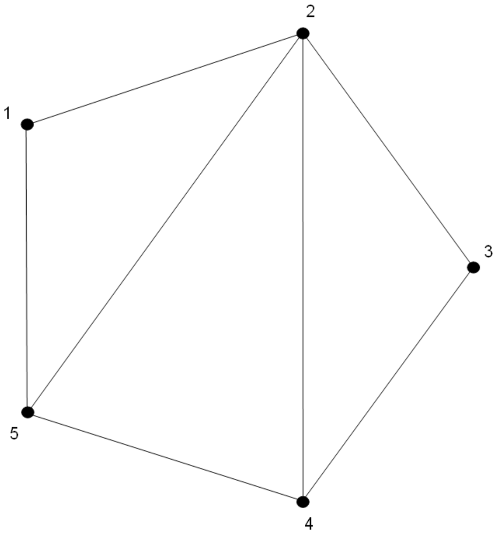

Fig.1. Graph associated with incomplete pairwise comparison matrix **C**

III. RESULTS

Theorem 1 [1]: The optimal solution of the problem (5) is unique if and only if the graph corresponding to the incomplete pairwise comparison matrix is connected.

Sketch of the proof: necessity is based on elementary linear algebra, sufficiency is more complex. By using the exponential parameterization $x_i = e^{y_i}$ and the notation $\mathbf{B}(\mathbf{y}) = \mathbf{B}(y_1,\ldots,y_d) = \mathbf{A}(x_1,\ldots,x_d) = \mathbf{A}(\mathbf{x})$, it follows from Kingman's Theorem that $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ is a logconvex (hence convex) function of $\mathbf{y}$. The next important step in the proof is that $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ is either strictly convex or constant along any line in the $\mathbf{y}$ space. Strict convexity in the case of the graph corresponding to the incomplete pairwise comparison matrix is connected comes from the lower bounds of $\lambda_{max}(\mathbf{A}(\mathbf{x}))$ expressed by the components of $\mathbf{x}$ and by the known entries of the incomplete matrix. It can be shown that if the minimum point of (5) is not unique then the graph is not connected.

By the next theorem, one may get a detailed view on the structure of the optimal set in the non-connected case

Theorem 2 [1]: The function $\lambda_{max}(\mathbf{B}(\mathbf{y}))$ attains its minimum and the optimal solutions constitute an $(s–1)$-dimensional affine set, where s is the number of the connected components in the graph corresponding to the incomplete pairwise comparison matrix.

Remark. The Perron eigenvalue of a pairwise comparison matrix is a non-convex function of its elements. Let **D** be a 3×3 pairwise comparison matrix of variable x as follows:

$$\mathbf{D} = \begin{array}{|c|c|c|} \hline 1 & 4 & x \\ \hline 1/4 & 1 & 2 \\ \hline 1/x & 1/2 & 1 \\ \hline \end{array}$$

The non-convex function $x \to \lambda_{max}(\mathbf{D}(x))$ is plotted in Fig. 2. However, by using the exponential scaling $x = e^t$, the function $t \to \lambda_{max}(\mathbf{D}(e^t))$ becomes a convex function (Fig. 3).

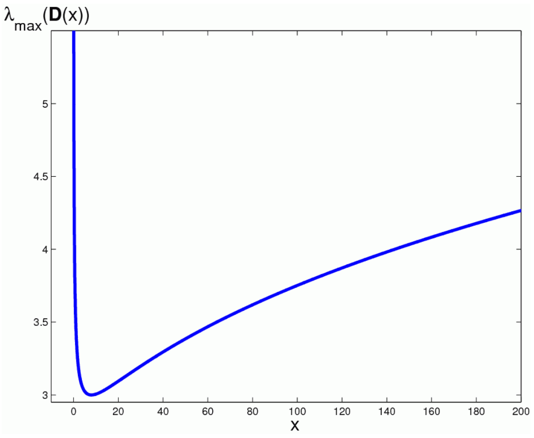

Fig.2. Non-convexity of function $x \to \lambda_{max}(\mathbf{D}(x))$

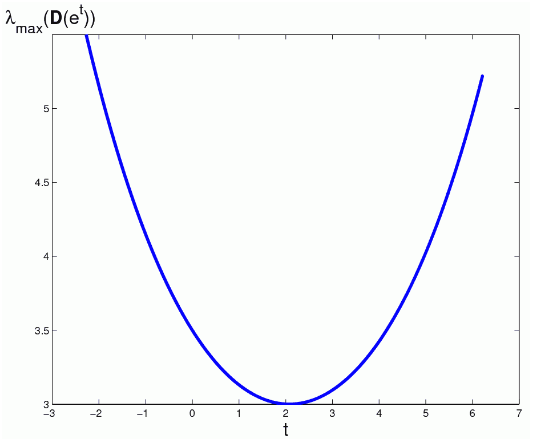

Fig.3. Convexity of function $t \to \lambda_{max}(\mathbf{D}(e^t))$

Theorem 3 [1]: The optimal solution of the incomplete *LLSM* problem (6) is unique if and only if graph corresponding to the incomplete pairwise comparison matrix is connected.

Sketch of the proof: Necessity follows from the definition of the problem. For sufficiency, taking the logarithms of the variables, and considering the first-order conditions of optimality, a linear system of equations is resulted in. It can be shown that the matrix of coefficients is strongly related to the Laplace matrix of the graph corresponding to the incomplete pairwise comparison matrix. The claim follows from the properties of the Laplace matrix regarding eigenvalues and rank.

The proof is constructive, one may directly apply it for a numerical example in order to compute the *LLSM* weights from an incomplete pairwise comparison matrix.

Remark: In the case of a complete pairwise comparison matrix, the solution of (6) results in the well-known geometrical mean.

An algorithm is proposed for solving the eigenvalue minimization problem (5) in the connected case [1]. Using the ideas of the proof of Theorem 1, the exponential parameterization of the variables is used, resulting in a strictly convex optimization problem. The latter can be solved, e.g., by the method of cyclic coordinates [9]. Moreover, the decision maker gets non-decreasing lower bound for the *CR*-inconsistency level in each step of the process of filling in the pairwise comparison matrix. This, especially in the case of a sharp jump, can be used for detecting misprints or false comparisons in real time.

Example. The solution of problem (5) for matrix **C** in (7), i.e., the $\lambda_{\max}$-optimal completion of **C**, and the corresponding *EM* weight vector, respectively, are as follows:

$$\mathbf{C}^{EM} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \mathbf{2.24} & \mathbf{5.41} & 8 \\ \hline 1/2 & 1 & 3 & 6 & 7 \\ \hline \mathbf{1/2.24} & 1/3 & 1 & 4 & \mathbf{5.67} \\ \hline \mathbf{1/5.41} & 1/6 & 1/4 & 1 & 5 \\ \hline 1/8 & 1/7 & \mathbf{1/5.67} & 1/5 & 1 \\ \hline \end{array} \qquad \mathbf{w}^{EM} = \begin{array}{|c|}\hline 0.392 \\ \hline 0.323 \\ \hline 0.177 \\ \hline 0.076 \\ \hline 0.032 \\ \hline \end{array}$$

The solution of problem (6) for matrix **C** in (7), i.e., the *LLSM* weights are as follows:

$$\mathbf{w}^{LLSM} = \begin{array}{|c|}\hline 0.404 \\ \hline 0.318 \\ \hline 0.174 \\ \hline 0.072 \\ \hline 0.032 \\ \hline \end{array}$$

Remark: Unlike the incomplete Eigenvector Method, the incomplete Logarithmic Least Squares Method does not use the completion of the matrix in order to compute the weights.

IV. DISCUSSION

From practical point of view, some important questions arise regarding incomplete pairwise comparison matrices and the application of Eigenvector Method for them. One common point of the questions is, e.g., how to use efficiently the time of the decision makers and keep all relevant information of their cardinal preferences. Another one is to provide a real-time feedback on the level of inconsistency, computed on the base of the already typed pairwise comparisons.

1. How many pairwise comparisons are *needed* in order to have an appropriate basis of decision maker's cardinal preferences? It follows from Theorem 1 and Theorem 2 that the minimal number of necessary comparisons is $n–1$, that equals to the number of edges of a spanning tree in a graph of n nodes. Since a graph on n nodes has at most $n(n–1)/2$ edges, this is the case of complete matrices. Consequently, the gap between the lower bound n and the upper bound $n(n–1)/2$ is large. It is an exciting topic of future research including real decision situations to give better approximations for the number of needed comparisons.

2. Which pairs to compare? If the number of needed comparisons is less than $n(n–1)/2$, which objects (criteria) should be compared to each other? There exist a large number of connected subgraphs of a complete graph of n nodes, one should choose of them based on experimental results from real decision problems again.

3. When to draw the decision maker's attention that the level of inconsistency sharply increased by the last typed pairwise comparison? What are the suitable thresholds? What to do if the level of inconsistency is too high (which is again to be defined) after a few comparisons and additional comparisons will make it only worse?

V. CONCLUSION

A natural necessary and sufficient condition, namely, the connectedness of the associated graph, is given for the uniqueness of the best completion of an incomplete pairwise comparison matrix regarding the Eigenvector Method and the Logarithmic Least Squares Method.

The eigenvalue optimization problem of the Eigenvector Method can be transformed into a convex, and, in the case of connected graph, into a strictly convex optimization problem, thus, weights can be computed from partial information. The decision maker gets a lower bound for the *CR*-inconsistency level in each step when filling in the pairwise comparison matrix, that can be used for detecting misprints in real time.

In the Logarithmic Least Squares problem for incomplete matrices, the geometric means of the rows' elements play important role in the explicit computation of the optimal solution, like in the complete case.

REFERENCES

[1] S. Bozóki, J. Fülöp, and L. Rónyai, “On optimal completions of incomplete pairwise comparison matrices,” *Working Paper 2009-1, Research Group of Operations Research and Decision Systems,Laboratory on Engineering and Management Intelligence,Computer and Automation Research Institute, Hungarian Academy of Sciences,* 30 pages, 2009. `http://www.sztaki.hu/~bozoki/BozokiFulopRonyai-WP-2009-1.pdf`

[2] B. Aupetit, and C. Genest, “On some useful properties of the Perron eigenvalue of a positive reciprocal matrix in the context of the analytic hierarchy process. ,” *European Journal of Operational Research*, vol. 70; pp. 263–268, 1993.

[3] G. Crawford G, and C. Williams, “A note on the analysis of subjective judgment matrices,” *Journal of Mathematical Psychology*, vol. 29, pp. 387–405, 1985.

[4] P. De Jong, “A statistical approach to Saaty's scaling methods for priorities,” *Journal of Mathematical Psychology*, vol. 28; pp. 467–478, 1984.

[5] P.T. Harker, “Incomplete pairwise comparisons in the analytic hierarchy process,” *Mathematical Modelling,* vol. 9, no. 11, pp. 837–848, 1987

[6] J.F.C. Kingman, “A convexity property of positive matrices,” *The Quarterly Journal of Mathematics*, vol. 12; pp. 283–284, 1961.

[7] M. Kwiesielewicz, “The logarithmic least squares and the generalised pseudoinverse in estimating ratios,” *European Journal of Operational Research*, vol. 93, pp. 611–619, 1996.

[8] M. Kwiesielewicz, and E. van Uden, “Ranking decision variants by subjective paired comparisons in cases with incomplete data,” in *Computational Science and Its Applications – ICCSA 2003, Lecture Notes in Computer Science,* Eds. V. Kumar et al., Springer-Verlag: Berlin/Heidelberg, 2669; pp. 208–215, 2003.

[9] D.G. Luenberger, and Y. Ye, *Linear and Nonlinear Programming*. Series: International Series in Operations Research & Management Science, vol. 116. 3rd Edition, Springer, 2008.

[10] O. Perron, “Zur Theorie der Matrizen,” *Mathematische Annalen* vol. 64, pp. 248–263, 1907.

[11] T.L. Saaty, *The analytic hierarchy process*. New York: McGraw-Hill, 1980.

[12] S. Shiraishi, T. Obata, and M. Daigo, “Properties of a positive reciprocal matrix and their application to AHP,” *Journal of the Operations Research Society of Japan*, vol. 41, no. 3, pp. 404–414, 1998.

[13] S. Shiraishi, and T. Obata, “On a maximization problem arising from a positive reciprocal matrix in AHP,” *Bulletin of Informatics and Cybernetics,* vol. 34, no. 2, pp. 91–96, 2002.

[14] E. van Uden, “Estimating missing data in pairwise comparison matrices,” in *Operational and Systems Research in the Face to Challenge the XXI Century, Methods and Techniques in Information Analysis and Decision Making,* Eds. Z. Bubnicki, O. Hryniewicz, and R. Kulikowski, Academic Printing House: Warsaw, pp. II-73–II-80, 2002.