Corvinus University of Budapest

Ph.D. Programme in Economics

WEIGHTS FROM PAIRWISE COMPARISONS AND EVALUATION BY USING UTILITY FUNCTIONS IN MULTI-ATTRIBUTE DECISION PROBLEMS

Summary of Ph.D. thesis

Sándor Bozóki

Budapest, 2006

Department of Decisions in Economics
located at the
Computer and Automation Research Institute,
Hungarian Academy of Sciences

Supervisor: Dr. Tamás Rapcsák

Contents

1 Multi-attribute decision problems

'Which one should I choose?' – this question comes to the surface every day and is an elementary momentum in human thinking and behavior. The amount of time and energy spent on finding the answer primarily depends on the importance of the problem. Solving problems of low importance is easy, a routine task, but decisions with serious consequences are preceded by consideration. My dissertation concentrates on the latter type of tasks: the importance of the problems requires thorough analysis.

The aim of a multi-attribute decision problem is to select the best one from among a given set of alternatives, with respect to certain attributes, or rank the alternatives.

Here, I will mention only some problems in Hungary, which received a wide publicity in 2004.

- tender on the concession building and operation of highway M6;
- the privatization of MALÉV Hungarian Airlines;
- the privatization of National Textbook Publishers;
- third generation mobile phones – tender;
- ordering trams to be run along the Grand Boulevard in Budapest;
- appointing the chairman of the MTV.

The examples from different fields share these common attributes:

– some of their attributes contradict each other;

– there is not a single best solution, mathematical;

– subjective factors influence the decision.

Some of the above problems resulted in serious political and social conflicts. Generally, the process of decision making is not public, therefore, it is difficult to judge from a professional point of view what turned some evaluations of certain tenders to scandals. By all accounts, the facts show that decision making on a professional basis is indeed necessary.

2 Utility functions

Utility functions, one of the basic concepts in economic sciences, play an important role in expected utility theory elaborated by Bernoulli [1], Ramsey [27] and Menger [26] and, similarly, in statistical decision theory by Wald [35]. The basic problem is as follows. Given a table containing possible actions (alternatives) $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, and possible occurrences $s_1, s_2, \ldots, s_k$ with probabilities $p_1, p_2, \ldots, p_k$.
$c_{11}, c_{12}, \ldots, c_{nk}$ – which can be called payments or winnings – represent the values of the alternatives on the basis of the occurrences.

Occurrences	s_1	s_2	$\ldots$	s_k
Probabilities of occurrences	p_1	p_2	$\cdots$	p_k
Alternatives				
$\mathcal{A}_1$	c_{11}	c_{12}	$\ldots$	c_{1k}
$\mathcal{A}_2$	c_{21}	c_{22}	$\ldots$	c_{2k}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathcal{A}_n$	c_{n1}	c_{n2}	$\ldots$	c_{nk}

If payments are considered through a utility function $u : \mathbb{R} \to \mathbb{R}$, then the aim is to select the j-th alternative with the maximal expected utility

$$\sum_{i=1}^{k} p_i u(c_{ji}).$$

The above is the basic problem of expected utility theory. The decision problem under uncertainty can be converted into a multi-attribute decision making problem as follows:

Decision making under uncertainty		Multi-attribute decision making
Actions (alternatives)	$\longleftrightarrow$	Alternatives
Occurrences	$\longleftrightarrow$	Attributes
Probabilities of the occurrences	$\longleftrightarrow$	Weights of attributes
Utility of payments	$\longleftrightarrow$	The evaluation of the alternatives, with respect to the attributes

The possibility of having the utility function in additive form and its application in multi-attribute decision problems were analyzed by Fishburn [14], Keeney and Raiffa [22]. A summary in Hungarian was given by Temesi [32].

The applications of utility functions in real decision problems are presented with specific examples in Sections 8 and 9. Some constructions of numerical utility functions are given in order to evaluate the alternatives with respect to the attributes, in solving practical problems converted into multi-attribute decision tasks.

3 Multi-attribute decision models

Modelling multi-attribute decision problems is a young discipline. Models of the past half-century may be classified mainly in three categories:

- basic methods – models based on elementary rules;
- methods aggregating, by using the weights of attributes, the evaluations of alternatives, with respect to the attributes;
- outranking methods.

Basic methods are based on concepts or heuristics that can be formulated easily. If the evaluation of the alternatives, with respect to az attribute, is given, one can select the best alternative by using a simple principle, or, at least, the range of alternatives gets narrower. In the lexicographic model, e.g., if the rank of importance of the attributes is known, then the winner will be the best alternative, with respect to the most important attribute. If there are several ones of this kind, then the second most important attribute is checked and the alternative(s) evaluated, with respect to it. The process is continued until there remains only one alternative.

In aggregating methods the solution of a multi-attribute decision problem consists of 3 main steps:

- evaluate the alternatives with respect to the attributes;
- determine the weights of importance of the attributes;
- aggregate the evaluations by using the weights of attributes.

Most cited models are Multi Attribute Utility/Value Theory (MAUT/MAVT, [22]), Analytic Hierarchy Process (AHP, [29]), and Simple Multi Attribute Ranking Technic (SMART, [9]).

The outranking relation was introduced by Roy [28] in order to compare the alternatives. An elementary step of the method is to decide how much an alternative is preferred to another one, with respect to an attribute. Methods ELECTRE [28], PROMETHEE [3], and KIPA [23] in the Hungarian relation, are the best known ones.

Weighting, i.e., the numerical expression of the importance of attributes, is needed in all models excluding basic methods. Weights reflect the goals and the preference of the decision makers. The difficulty is that importance has no generally accepted unit of measure, it can only be interpreted together with some sort of scale. It is possible that the decision maker can specify the weights of attributes directly in a numerical way. This process is also called simple direct weighting. In case of complex problems the decision maker cannot be requested to give all the weights of attributes converted into numbers to the modeler. The division of the decision problem into smaller parts allows the decision maker to answer simple and clear questions, from which the weights of the whole problem can be computed.

Some of the best known weighting methods are the simple direct weighting, mentioned above, methods of Churchman-Ackoff and Guilford, linear programming techniques, the SMART, and methods based on pairwise comparisons. A class of the latter is discussed in the dissertation.

We assume that the premise used in preference modelling is an axiom, according to which decision makers are able to compare two objects (e.g., the importance of two attributes) and tell that one is better (or bigger) than the other or they are equal.

In the 1780's, Condorcet [6] and Borda [2] introduced the concept of *pairwise comparison* as a relation between two elements of a rank built based on individual preferences. Pairwise comparisons were used in experimental psychology by Weber and Fechner [13] in the middle of the 19th century, followed by Thorndike [33] and Thurstone [34] in the 1920's. The applications of pairwise comparisons were studied in a historical and methodological summary by Kindler and Papp [23].

In the dissertation a class of pairwise comparisons is discussed, in which elements are compared in a ratio scale. The decision maker is requested to compare the objects as follows: how many times one is *better* or *bigger* than the other one [29]. The object to be compared depends on the problem and may appear as:

- the importance of attributes;
- the evaluations of alternatives, with respect to the attributes;
- weights of competence, in group decision making.

Pairwise comparisons can be arranged into a square matrix. The definition is as follows.

Definition. *Let $\mathbb{R}_+^{n\times n}$ denote the class of $n \times n$ matrices containing positive elements. The matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & a_{2n} \\ 1/a_{13} & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & 1/a_{3n} & \dots & 1 \end{pmatrix} \in \mathbb{R}_+^{n\times n}$$

*is called **pairwise comparison matrix** if*

$$a_{ii} = 1, \tag{1}$$

$$a_{ij} = \frac{1}{a_{ji}} \tag{2}$$

for all indices $i, j = 1, \dots, n$.

a_{ij} shows how many times the i-th element is judged to be better than the j-th one by the decision maker. According to equation (1), a comparison of each element to itself results in 1.
Property (2) is based on the reciprocity assumption, i.e. if the i-th element is a_{ij} times bigger than the j-th one, then the j-th one is $\frac{1}{a_{ij}}$ times bigger than the i-th one. Considering properties (1)-(2), a pairwise comparison matrix referring to n elements can be written by using $\binom{n}{2} = \frac{n(n-1)}{2}$ comparisons.

Definition. *A matrix $\mathbf{A} = [a_{ij}]_{i,j=1,2,\dots,n} \in \mathbb{R}_+^{n\times n}$ of properties* (1)-(2) *satisfying*

$$a_{ij}a_{jk} = a_{ik} \tag{3}$$

*for all indices $i, j, k = 1, \dots, n$, is called **consistent** pairwise comparison matrix. A matrix satisfying properties* (1)-(2) *but violating* (3) is called **inconsistent**.

The aim is to determine weights $w_1, w_2, \dots, w_n$ from matrix $\mathbf{A}$ containing the pairwise comparisons of the elements. Conditions referring to the weights are as follows:

$$w_i > 0, \qquad i = 1, 2, \dots, n, \tag{4}$$

$$\sum_{i=1}^{n} w_i = 1. \tag{5}$$

The weights are jointly denoted by weight vector $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$.

Several methods exist for solving the weighting problem. In Analytic Hierarchy Process (AHP, [29]), the weight vector is computed as the normalized right eigenvector corresponding to the maximal eigenvalue of matrix **A**. In each distance minimizing method, an objective function is considered and the weights are computed from the optimal solution(s). The Least Squares Method [5] and its relaxed modifications, e.g., Weighted Least Squares, Logarithmic Least Squares, and Chi Squares, and also the Singular Value Decomposition [17] and Goal Programming are the best known weighting methods.

In case of a consistent matrix, all the methods result in the same weight vector. The difference of results arises when the matrix is inconsistent. A multi-attribute analysis was done by Golany and Kress [18]. They concluded that each method has advantages as well as drawbacks, therefore, none of them is 'prime'.

The Least Squares Method (LSM) problem was defined more than 30 years ago, even so, I found rather few papers in solving it. Unlike other methods, the LSM may have multiple solutions [20]. The objective function may be non-convex and the difficulty of using algorithms based on Newton's iteration ([20], [11]) is to find all the optima. Solutions usually depend on the initial point. I have not found a paper in computing all the local and global optima of LSM until now.

The aim of my research was to analyze and solve problems, according to my knowledge have not been studied by others, as follows:

- computing all the solutions of the LSM problem corresponding to pairwise comparison matrices;
- analyzing the structure of a pairwise comparison matrix in view of all the solutions ;
- studying the consequences of the existence of multiple solutions in decision theoretical context, e.g., referring to the inconsistency of a decision maker or the pairwise comparison matrix filled in by her/him.

The main application area of weighting methods is to determine the weights of attributes in multi-attribute decision problems, whereas, it can be used for the evaluation of the alternatives or voting powers of decision makers in group decision problems.

It is also shown in the dissertation that the applicability of LSM is not restricted to the matrices completely filled in (containing $\binom{n}{2}$ comparisons). In this point of view, it is more general, then, e.g., the Eigenvector Method.

4 Methods used in research

Solving polynomial systems

The minimization problem of LSM is approached by a transformation of the first-order optimality conditions into a multivariate polynomial system. Polynomial systems are often used in mathematical (geometrical), physical and engineering (kinetic and equilibrium) problems. The solution, just like in the case of non-linear systems, is not simple. The four methods below are studied for solving tasks of small size:

- resultant method;
- Gröbner-basis;
- generalized resultant method;
- homotopy algorithm.

Since all the solutions of a polynomial system are required, methods based on Newton-iteration are not considered here. However, if an approximating solution computed by using a polynomial system solver method is chosen as an initial value of the Newton-iteration, it is possible to attain arbitrary exactness.

4.1 Resultant method

The resultant is originated from the roots of two univariate polynomials. It indicates whether the polynomials have a common root. If so, the resultant is equal to zero. In other words, it is a necessary condition of the existence of a common root. The solution of a system of two polynomials of two variables can be written as the roots of a univariate polynomial. The theoretic elegancy of resultant method meets with limits in practical use. As I found, systems of more than 2 equations cannot be solved with the resultant method.

4.2 Gröbner-basis

Gröbner-basis was defined by Buchberger [4] for analysing polynomial rings and ideals. A Gröbner-basis corresponding to a polynomial system is equivalent to the original one, in the sense that they have the same roots. Nevertheless, Gröbner-basis have additional properties, which may be made use of during the division by polynomials and other operations.

4.3 Generalized resultant

The generalized resultant was introduced by Dixon [7] (and is called Dixon-resultant after him) for solving polynomial systems of two or more equations. The role of Dixon-resultant is the same as the resultant's: it is computed from the coefficients of the multivariate polynomials, and is equal to zero if and only if the polynomial system has a solution (common root). The algorithm based on Dixon-resultant proposed by Bezout, Dixon, Kapur, Saxena and Yang [21] was implemented by Lewis [24] in computer algebra system Fermat, which was also developed by him.

4.4 Homotopy algorithm

Homotopy continuity methods have been developed in the past 25 years, and now they are considered as reliable and efficient algorithms for computing all the solutions of non-linear systems.

The numerical computation of all the solutions of polynomial systems by homotopy algorithm was first proposed by Garcia, Zangwill [16], and independently, by Drexler [8].

I have used the algorithm of Li and Gao [25, 15] for the computation of the solutions of polynomial systems corresponding to the LSM, as special non-linear systems.

5 Theoretical and methodological results

5.1 Solution of the Least Squares Method (LSM) problem

The problem of LSM, one of the weighting methods based on pairwise comparison matrices, was solved in Section 4 regarding the sizes of $3\times3, 4\times4, \ldots, 8\times8$. Since the non-linear objective function to be minimized is nonconvex, the optimum is usually not unique. The methods based on Newton iteration, applied formerly to the solution of the problem, are of the specific feature that the solution is sensitive regarding the initial point. The methods discussed in the dissertation are able to find all the local and global minima. By my experience, the resultant method and Gröbner-bases may be applied to 3×3 matrices, the software Fermat, based on generalized resultants, to 3×3 and 4×4 matrices, and the homotopy method to $3 \times 3, \ldots, 8 \times 8$ matrices.

Individual results:

- transforming the least squares optimization problem into finding all the common roots of a multivariate polynomial system [P-1];
- implementing the resultant method in softwares Maple and Matlab in the case of 3×3 matrices;
- constructing the polynomial system corresponding to the LSM-problem for any size of matrices [P-3].

Results from joint work:

- the solution of the polynomial system of 3 variables and 3 equations corresponding to 4×4 matrices, by using the software Fermat implemented by Lewis [P-2];
- the solution of the polynomial systems deriving from $3 \times 3, 4 \times 4,$ $\ldots, 8 \times 8$ matrices, by using homotopy method implemented by Gao [P-3].

5.2 Numerical results

Individual results:

At the present stage of research, I can generate 3×3 matrices in large numbers and compute LSM-weights automatically. Comparing the Eigenvector, Least Squares and Singular Value Decomposition Methods, I concluded as follows:

- the SVD-inconsistency significally differs from the other two. In the almost consistent range, the SVD-inconsistency is the best in distinguishing the different levels of a decision maker's inconsequence;
- matrices acceptable by the 10%-rule of EM-inconsistency may be approximated in LSM sense, too, with a small error;
- examples were presented for matrices of less than 10% EM-inconsistency but the SVD-inconsistency is high;
- as the level of inconsistency increases, the weight vectors computed by various methods differ from each other more and more;
- in case of large EM-inconsistency, the EM weight vector is close to the vector of equal components, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the LSM-solution is, most often, not unique.

I examined the (CR) value of EM-inconsistency, used for measuring the level of inconsistency of pairwise comparison matrices, by a statistical analysis of a large number of randomly generated matrices. As numerical results show, the 10% rule of EM-inconsistency is significally different regarding various sizes of the matrix:

- for $n = 3$, a significant share (28%) of matrices is acceptable;
- for $n = 4, 5$, a small share is acceptable;
- for $n = 6, 7$, few matrices are acceptable;
- for $n = 8, 9, 10$, no matrix of acceptable inconsistency was found in the sample of ten million randomly generated matrices.

The aim is of the actual form of the result above is to raise further questions. Does a relation exist, (and in the affirmative, what kind?) between the levels of inconsistency of a concrete pairwise comparison matrix filled in by a decision maker faced with a concrete problem in a real decision situation and the matrices generated randomly? For the answers, further research analysing matrices from practice is needed.

5.3 Research directions

At present, the computation of LSM-weights of $4 \times 4, \ldots, 8 \times 8$ matrices is done one by one, therefore, the possibility of statistical analysis is limited. CPU-times (especially in cases of $n = 7, 8$) point that methods discussed in the dissertation are still not suitable for solving decision problems in real time. Algorithms are in stage of research.

Present algorithms and computational capacities allow us to do calculations with matrices of maximum size 8×8. In complex problems comparing more than 8 objects (e.g., importance of attributes) might be necessary. The LSM-problem regarding 9×9 and 10×10 matrices is still unsolved.

When solving decision problems, it is necessary to guarantee that the weight vector computed from a pairwise comparison matrix be unique. The property of uniqueness holds for the Eigenvalue Method and the Singular Value Decomposition Method but a necessary and sufficient condition for the Least Squares Method is still unknown. A necessary condition of the non-uniqueness of the least squares solution of a class of pairwise comparison matrices was given by Farkas and Rózsa [12].

A comparison of weighting methods is also essential both in decision theory and in practical applications. The aim is to select the method suitable for specific properties of decision problems the best. The mapping and identifying of properties is a subject of present research.

The Least Squares Method was originally defined for matrices completely filled in. But it can be written also in case of missing elements. In practice, the time of decision makers is a bottleneck, therefore, techniques are needed for getting enough information as efficiently as possible. First of all, matrices of larger sizes such as $8 \times 8, 9 \times 9, 10 \times 10$ may be interesting from this point of view. From the computations of the first stage of my research, I concluded that in some cases it is enough to have essentially fewer comparisons than $\frac{n(n-1)}{2}$.

6 Pairwise comparison matrix in Leontief input-output model

Pairwise comparison matrices are related not only to multi-attribute decision problems, but other areas as well. One of them is the dynamic Leontief input-output model in case of balanced growth [36].

Stojanović [31] showed that in case of an economy of balanced growth, in which each sector grows at the same rate, the *growth matrix,* defined by himself, can be written as a scalar multiplication of a pairwise comparison matrix.

Steenge [30] proved that both static and dynamic-stationary Leontief input-output model can be written with a pairwise comparison matrix.

Research directions

In practice, the growth matrix constructed from the empirical growth rates of the sectors does not satisfy all the conditions of the models. Conclusions might be drawn for the model as a dynamic system by mapping the eigenvalues of the matrix, as Farkas and Rózsa [10] analysed a class of specially perturbed pairwise comparison matrices.

7 Application: Ranking bank projects

The application discussed in this section was ordered by the center of an international bank in Hungary (mentioned as 'Bank' in the following parts). The Bank faced the problem of ranking 50-100 projects. Our aim was to construct a model for the prioritization and the recommendation of implementation, which works also in the case of a dinamically changing set of alternatives.
The work has been completed by the Laboratory and Department of Operations Research and Decision Systems, Computer and Automation Research Institute, Hungarian Academy of Sciences in 2001/02 [CS-1].

We did not find a decision model in the literature which could have been directly adapted to solve the problem, consequently, we created a new model. The attributes defined together with experts from the Bank were organized in a tree structure. The weights of the attributes were computed based on the pairwise comparison matrices filled in by the top management of the Bank, which made it possible to adapt the ranking model to the Bank's strategy.

The construction of utility functions based on the information from the Bank was recommended for the evaluation of projects (alternatives). In case of objective attributes, the evaluation was done automatically, by using built-in utility functions. For subjective attributes, uniform and clear-cut scales were introduced in order to make the work of the decision makers participating in the evaluation process easier.

Our model was installed in the Bank in 2002 and, according to a written reference, is still running successfully.

Individual results:

- **computation of the weights of attributes** from pairwise comparison matrices given by the top management of the Bank;
- building a method for the evaluation of the alternatives, **construction of utility functions** based on the instructions given by the Bank's representative;
- delimitation of the applicability of Expert Choice in this problem.

8 Model: Decision tasks in *Brainfarm*

The last section of the dissertation presents a modelling task in the project *Brainfarm,* which is a collaborative model for academic communication, publication and research on-line. The aims of this task were to design a recommendation system, explore the system of connections among users, formulate and solve the process of group formation as a decision problem. [CS-2]. *Brainfarm* has been engineered by a co-operation of Media Research Centre, Department of Sociology, Budapest University of Technology and Economics along with the Laboratory and Department of Operations Research and Decision Systems, Computer and Automation Research Institute, Hungarian Academy of Sciences and Frutta Elettronica, a multimedia development company.

Our team constructed decision models, new ones in the international literature, by which tasks as follows can be handled:

- real time recommendation in a system of thousands of users and several ten thousands of pages and documents;
- the follow-up of the users' activity and its feedback into the system;
- the creation and strengthening of connections among users;
- decision situations during the operation.

Individual results:

- detecting and modelling tasks which may be defined as decision problems;
- defining the attributes of multi-attribute decision problems;
- **constructing methods (utility functions) for evaluation, with respect to the attributes**;
- **computing the weights of attributes based on pairwise comparison matrices**;
- summarizing some methods for measuring the similarity of user profiles and evaluations in *Brainfarm*.

9 Main references

[1] Bernoulli, D. [1738]: Specimen theoriae novae de mensura sortis, *Commentarii Academiae Scientiarum Imperalis Petropolitanae,* in English: Exposition of a new theory of measurement of risk, *Econometrica* **51**(4), July, pp. 1065-1092.

[2] Borda, J.C. de [1781]: Mémoire sur les électiones au scrutin, *Histoire de l'Académie Royale des Sciences,* Paris.

[3] Brans, J.P. [1982]: L'ingéniérie de la décision, Élaboratorion d'instruments d'aide ŕ la décision. Méthode PROMETHEE, Université Laval, Collogue d'Aide ŕ la Décision Québec Canada, pp. 183-213.

[4] Buchberger, B. [1985]: An algorithmic method in polynomial ideal theory, in: Multidimensional System Theory, N.K. Bose (editor), *D. Rediel Publishing Company,* Dordrecht, Boston, Lancaster, pp. 184-232.

[5] Chu, A.T.W., Kalaba, R.E., Spingarn, K. [1979]: A comparison of two methods for determining the weight belonging to fuzzy sets, *Journal of Optimization Theory and Applications,* **4**, pp. 531-538.

[6] Condorcet, M. [1785]: Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues á la Pluralité des Voix, Paris.

[7] Dixon, A.L. [1908]: The eliminant of three quantities in two independent variables, *Proceedings of the London Mathematical Society,* **7**, pp. 50-69, pp. 473-492.

[8] Drexler, F.J. [1978]: Eine Methode zur Berechnung sämtlicher Lösungen von Polynomgleichungssystemen, *Numerische Mathematik,* **29**, pp. 45-58.

[9] Edwards, W. [1977]: How to use multiattribute utility measurement for social decision making, *IEEE Transactions on Systems, Man, and Cybernetics,* **SMC-7**, **5**, pp. 326-340.

[10] Farkas, A., Rózsa, P. [2001]: Data Perturbations of Matrices of Pairwise Comparisons, *Annals of Operations Research,* **101**, pp. 401-425.

[11] Farkas, A., Lancaster, P., Rózsa, P. [2003]: Consistency adjustment for pairwise comparison matrices, *Numerical Linear Algebra with Applications,* **10**, pp. 689-700.

[12] Farkas, A., Rózsa, P. [2004]: On the Non-Uniqueness of the Solution to the Least-Squares Optimization of Pairwsie Comparison Matrices, *Acta Polytechnica Hungarica, Journal of Applied Sciences at Budapest Polytechnic Hungary,* **1**, pp. 1-20.

[13] Fechner, G.T. [1860]: Elemente der Psychophysik, *Breitkopf und Härtel,* Leipzig.

[14] Fishburn, P.C. [1970]: Utility Theory for Decision Making, *Wiley,* New York.

[15] Gao, T., Li, T.Y., Wang, X. [1999]: Finding isolated zeros of polynomial systems in $\mathbb{C}^n$ with stable mixed volumes, *Journal of Symbolic Computation,* **28**, pp. 187-211.

[16] Garcia, C.B., Zangwill, W.I. [1979]: Finding all solutions to polynomial systems and other systems of equations, *Mathematical Programming,* **16**, pp. 159-176.

[17] Gass, S.I., Rapcsák, T. [2004]: Singular value decomposition in AHP, *European Journal of Operations Research,* **154**, pp. 573-584.

[18] Golany, B., Kress, M. [1993]: A multicriteria evaluation of methods for obtaining weights from ratio-scale matrices, *European Journal of Operations Research,* **69**, pp. 210-220.

[19] Guilford, J.P. [1936]: Psychometric Methods, *McGraw-Hill Book,* New York.

[20] Jensen, R.E. [1984]: An Alternative Scaling Method for Priorities in Hierarchical Structures, *Journal of Mathematical Psychology,* **28**, pp. 317-332.

[21] Kapur, D., Saxena, T., Yang, L. [1994]: Algebraic and geometric reasoning using Dixon resultants. In: *Proceedings of the International Symposium on Symbolic and Algebraic Computation.* A.C.M. Press.

[22] Keeney, R.L., Raiffa, H. [1976]: Decisions with Multiple Objectives: Preferences and Value Tradeoffs, *John Wiley* & *Sons*, New York.

[23] Kindler J., Papp, O. [1977]: Komplex rendszerek vizsgálata – Összemérési módszerek (in Hungarian), *Műszaki Könyvkiadó,* Budapest.

[24] Lewis, R.H. : Computer algebra system *Fermat.*
http://www.bway.net/˜lewis/

[25] Li, T.Y. [1997]: Numerical solution of multivariate polynomial systems by homotopy continuation methods, *Acta Numerica,* **6**, pp. 399-436.

[26] Menger, K. [1934]: Das Unsicherheitsmoment in der Wertlehre. Betrachtungen in Anschluss an das sogenannte Petersburger Spiel, *Zeitschrift für Nationalökonomie,* **5**, pp. 459-485.

[27] Ramsey, F.P. [1931]: Truth and probability, in: Ramsey, F.P.: The Foundations of Mathematics and other Logical Essays, ed. R.B. Braithwaite, *Kegan Paul, Trench, Trubner and Co.,* London.

[28] Roy, Bernard [1969]: Algebre Moderne et Theorie des Graphes Orientees vers les Sciences Economiques et Sociales, Paris, Dunod.

[29] Saaty, T.L. [1980]: The analytic hierarchy process, *McGraw-Hill,* New York.

[30] Steenge, A.E. [1981]: The Verification of Efficient Growth: An Approach via Stojanović's Matrix of Growth, *Journal of Macroeconomics,* **3**, No. 2, pp. 271-281.

[31] Stojanović, D. [1974]: Model kretanja privede sap vojvodine na bazi matrice rasta, (English Summary), *Mathematika Balkanika,* **4**.

[32] Temesi, J. [2002]: A döntéselmélet alapjai (in Hungarian), *Aula Kiadó,* Budapest.

[33] Thorndike, E.L. [1920]: A Constant Error in Psychological Ratings, *Journal of Applied Psychology,* **4**, pp. 25-29.

[34] Thurstone, L.L. [1927]: The Method of Paired Comparisons for Social Values, *Journal of Abnormal and Social Psychology,* **21**, pp. 384-400.

[35] Wald, A. [1950]: Statistical Decision Functions, *Wiley,* New York.

[36] Zalai, E. [2000]: Matematikai közgazdaságtan (In Hungarian), *KJK-KERSZÖV,* Budapest.

10 Publications and case studies

[P-1] Bozóki, S. [2003]: A method for solving *LSM* problems of small size in the AHP, *Central European Journal of Operations Research,* **11**, pp. 17-33.

[P-2] Bozóki, S., Lewis, R.H. [2005]: Solving the Least Squares Method problem in the AHP for 3×3 and 4×4 matrices, *Central European Journal of Operations Research,* **13**, pp. 255-270.

[P-3] Bozóki S. [2005]: Súlyok meghatározása páros összehasonlítás mátrixok legkisebb négyzetes közelítése alapján (in Hungarian), *Alkalmazott Matematikai Lapok* (to appear).

[CS-1] Rapcsák, T., Bozóki, S. [2001]: Ranking bank projects, Department of Operations Research and Decision Systems, Computer and Automation Research Institute, Hungarian Academy of Sciences.

[CS-2] Rapcsák, T., Bozóki, S., Lakatos, V., Selmeczy, I. [2003]: Decision tasks in Brainfarm, Department of Operations Research and Decision Systems, Computer and Automation Research Institute, Hungarian Academy of Sciences.