

Eigenvalue optimization by Newton's method for incomplete pairwise comparison matrices

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Outline

- Pairwise comparison matrix
- Incomplete pairwise comparison matrix
- Eigenvalue optimization
- Cyclic coordinates
- Newton's method in one variable
- Newton's method in higher dimensions

Given n objects with weights $w_1, w_2, w_3, \dots, w_n$. The **pairwise comparison matrix** is defined as follows:

$$\begin{pmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \frac{w_2}{w_3} & \cdots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & 1 & \cdots & \frac{w_3}{w_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \cdots & 1 \end{pmatrix},$$

where

$$w_{ij} > 0,$$

$$w_{ij} = \frac{1}{w_{ji}},$$

$$w_{ij} = w_{ik}w_{kj}.$$

for any i, j, k indices.

In real decision situations, weights are unknown, but pairwise comparisons can be made:

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix},$$

where

$$a_{ij} > 0,$$
$$a_{ij} = \frac{1}{a_{ji}}.$$

for $i, j = 1, \dots, n$. The aim is to determine the weight vector $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{R}_+^n$.

In the **Eigenvector Method (EM)** the approximation \mathbf{w}^{EM} of \mathbf{w} is defined by

$$\mathbf{A}\mathbf{w}^{EM} = \lambda_{max}\mathbf{w}^{EM},$$

where λ_{max} denotes the maximal eigenvalue, also known as Perron eigenvalue, of \mathbf{A} and \mathbf{w}^{EM} denotes the the right-hand side eigenvector of \mathbf{A} corresponding to λ_{max} .

By Perron's theorem, \mathbf{w}^{EM} is positive and unique up to a scalar multiplication. The most often used normalization is

$$\sum_{i=1}^n w_i^{EM} = 1.$$

Saaty defined the inconsistency ratio as $CR = \frac{\lambda_{max} - n}{RI_n}$, where λ_{max} is the Perron eigenvalue of the complete pairwise comparison matrix given by the decision maker, and RI_n is defined as $\frac{\overline{\lambda_{max} - n}}{n-1}$, where $\overline{\lambda_{max}}$ is an average value of the Perron eigenvalues of randomly generated $n \times n$ pairwise comparison matrices.

It is well known that $\lambda_{max} \geq n$ and equals to n if and only if the matrix is consistent, i.e., the transitivity property holds. It follows from the definition that CR is a positive linear transformation of λ_{max} .

According to Saaty, larger value of CR indicates higher level of inconsistency and the 10%-rule ($CR \leq 0.10$) separates acceptable matrices from unacceptable ones.

Incomplete pairwise comparison matrix
 (= pairwise comparison matrix with missing elements)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & - & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & - \\ - & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & - & 1/a_{3n} & \dots & 1 \end{pmatrix} .$$

Incomplete pairwise comparison matrix
 (= pairwise comparison matrix with missing elements)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & - \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & - & 1/a_{3n} & \dots & 1 \end{pmatrix} .$$

Incomplete pairwise comparison matrix
 (= pairwise comparison matrix with missing elements)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & x_d \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/x_d & 1/a_{3n} & \dots & 1 \end{pmatrix},$$

Incomplete pairwise comparison matrix
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$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & x_d \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/x_d & 1/a_{3n} & \dots & 1 \end{pmatrix},$$

where $x_1, x_2, \dots, x_d \in \mathbb{R}_+$.

Based on the idea above, Shiraishi, Obata and Daigo considered the eigenvalue optimization problems as follows.

In case of one missing element, denoted by x , the $\lambda_{max}(\mathbf{A}(x))$ to be minimized:

$$\min_{x>0} \lambda_{max}(\mathbf{A}(x)).$$

In case of more than one missing elements, arranged in vector \mathbf{x} , the aim is to solve

$$\min_{\mathbf{x}>\mathbf{0}} \lambda_{max}(\mathbf{A}(\mathbf{x})).$$

Graph representation of a pairwise comparison matrix

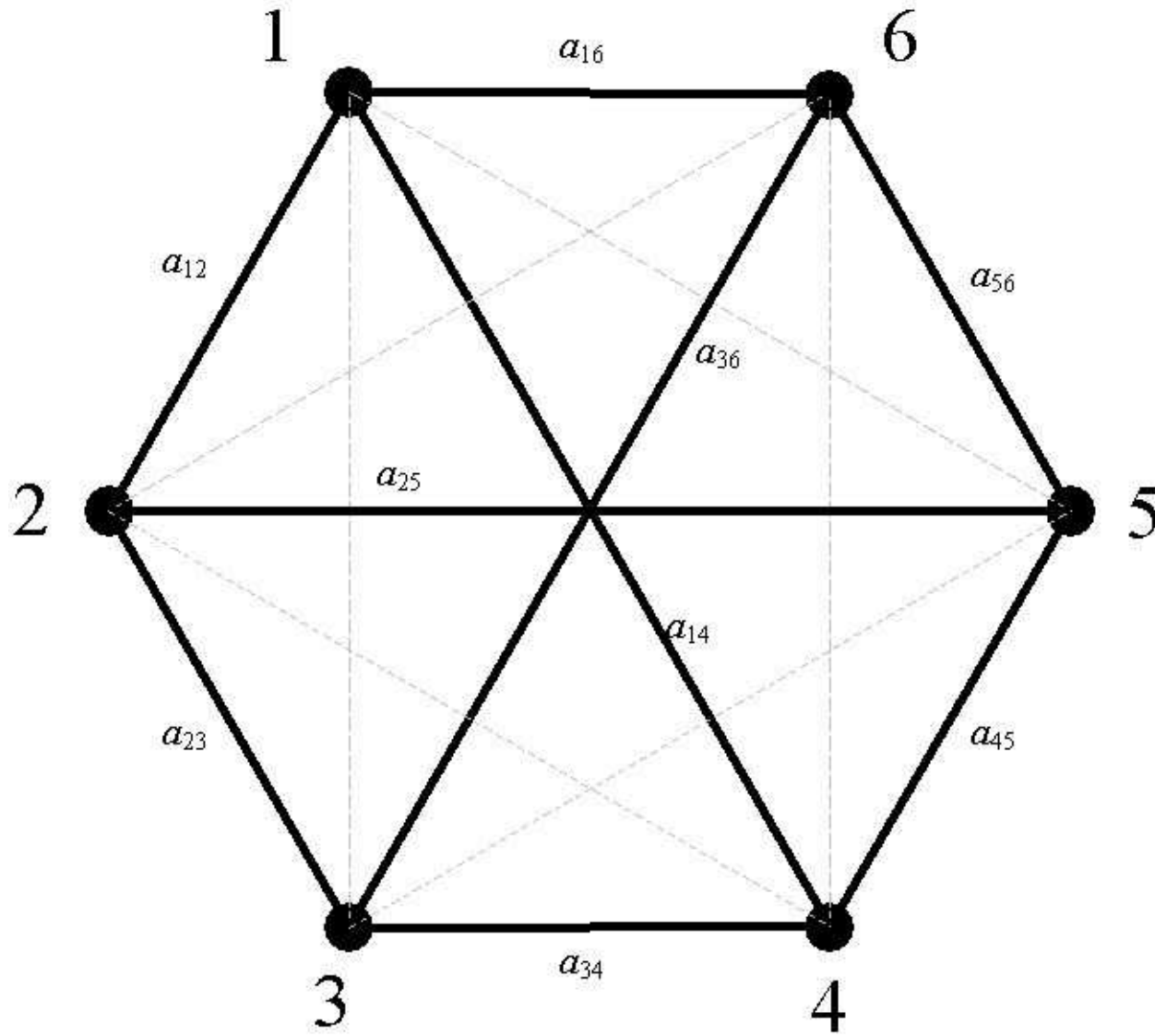
Given A incomplete pairwise comparison matrix of size $n \times n$. Graph $G = (V, E)$ is defined as follows:

$$V = \{1, 2, \dots, n\}$$

$$E = \{e(i, j) \mid a_{ij} \text{ (and } a_{ji}) \text{ are given and } i \neq j\}$$

Special case: all the comparisons are given, the corresponding graph is K_n .

Graph representation of a pairwise comparison matrix



	1	2	3	4	5	6
1	-	a_{12}	-	a_{14}	-	a_{16}
2	a_{21}	-	a_{23}	-	a_{25}	-
3	-	a_{32}	-	a_{34}	-	a_{36}
4	a_{41}	-	a_{43}	-	a_{45}	-
5	-	a_{52}	-	a_{54}	-	a_{56}
6	a_{61}	-	a_{63}	-	a_{65}	-

Theorem (B., Fülöp, Rónyai, 2010): The optimal solution of the eigenvalue minimization problem

$$\min_{\mathbf{x} > \mathbf{0}} \lambda_{max}(\mathbf{A}(\mathbf{x})).$$

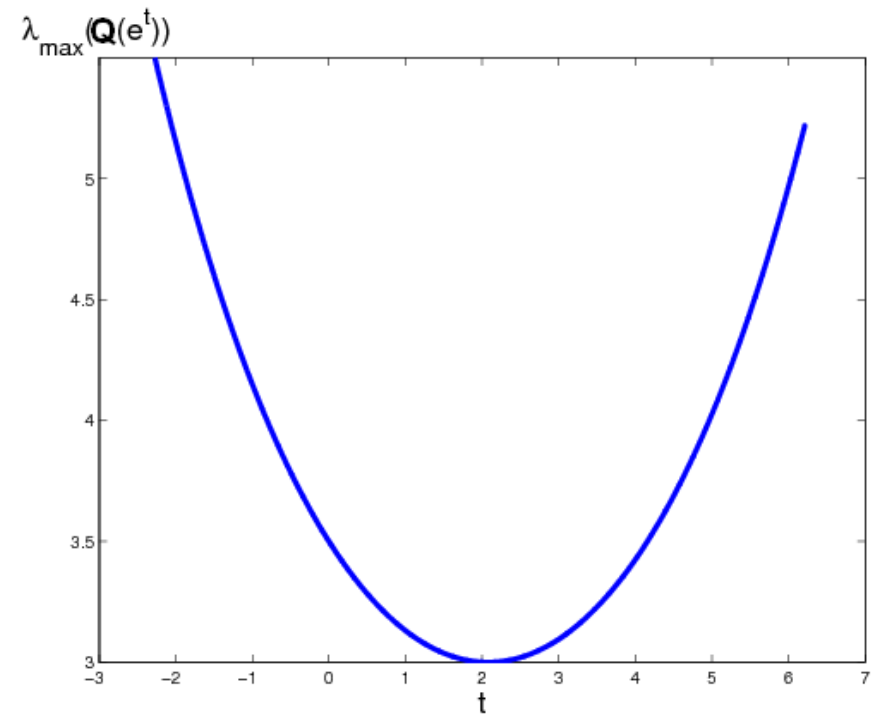
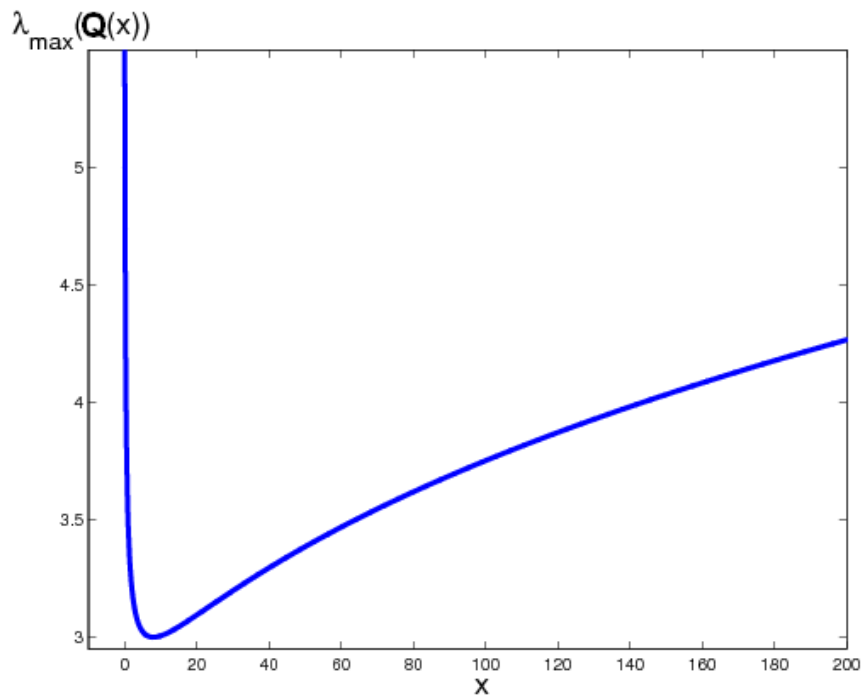
is unique if and only if the graph G corresponding to the incomplete pairwise comparison matrix is *connected*.

If graph G corresponding to the incomplete pairwise comparison matrix is connected, then by using the exponential parametrization $x_1 = e^{y_1}, x_2 = e^{y_2}, \dots, x_d = e^{y_d}$, the eigenvalue minimization problem is transformed into a strictly convex optimization problem.

Example

$$\mathbf{Q} = \begin{pmatrix} 1 & 2 & x \\ 1/2 & 1 & 4 \\ 1/x & 1/4 & 1 \end{pmatrix}.$$

$\lambda_{\max}(\mathbf{Q}(x))$ and, by using the exponential scaling $x = e^t$,
 $\lambda_{\max}(\mathbf{Q}(e^t))$ are plotted.



Algorithms for solving the eigenvalue minimization problem

$$\min_{\mathbf{x} > \mathbf{0}} \lambda_{max}(\mathbf{A}(\mathbf{x})).$$

- cyclic coordinates with Matlab's function *fminbnd*
- cyclic coordinates univariate Newton's method
- multivariate Newton's method

Method of cyclic coordinates

$$\mathbf{M}(\mathbf{x}) = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & x_1 & 5 & x_2 & 3 & x_3 & 1/7 \\ 1/3 & 1/x_1 & 1 & x_4 & 3 & x_5 & 6 & x_6 \\ 1/7 & 1/5 & 1/x_4 & 1 & x_7 & 1/4 & x_8 & 1/8 \\ 1/6 & 1/x_2 & 1/3 & 1/x_7 & 1 & x_9 & 1/5 & x_{10} \\ 1/6 & 1/3 & 1/x_5 & 4 & 1/x_9 & 1 & x_{11} & 1/6 \\ 3 & 1/x_3 & 1/6 & 1/x_8 & 5 & 1/x_{11} & 1 & x_{12} \\ 4 & 7 & 1/x_6 & 8 & 1/x_{10} & 6 & 1/x_{12} & 1 \end{pmatrix}$$

Method of cyclic coordinates

Let $x_i^{(k)}$ denote the value of x_i in the k -th step of the iteration, which has d (in the example, $d = 12$) substeps for each k .

For $k = 0$:

Let the initial points be equal to 1 for every variable:

$$x_i^{(0)} := 1 \quad (i = 1, 2, \dots, d).$$

while $\max_{i=1,2,\dots,d} \|x_i^k - x_i^{k-1}\| > T$

$x_i^{(k)} :=$

$\arg \min_{x_i} \lambda_{max}(\mathbf{M}(x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k-1)}, \dots, x_d^{(k-1)})), i =$

$1, 2, \dots, d$

next k

end while

Method of cyclic coordinates

Focus on $\min_{x_i} \lambda_{max}(\mathbf{M}(x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k-1)}, \dots, x_d^{(k-1)}))$

Matlab's function *fminbnd* solves is directly and fast.

Univariate Newton's method can be also applied.

$$\min_{x>0} \lambda_{max}(\mathbf{A}(x))$$

Let $x = e^t$ and $L(t) = \lambda_{max}(e^t)$.

$$t_{n+1} = t_n - \frac{L'(t_n)}{L''(t_n)} = t_n - \frac{\frac{\partial \lambda_{max}(x)}{\partial x}}{\frac{\partial^2 \lambda_{max}(x)}{(\partial x)^2} \cdot e^{t_n} + \frac{\partial \lambda_{max}(x)}{\partial x}}.$$

By Harker, formal derivatives $\frac{\partial \lambda_{max}(x)}{\partial x}$ and $\frac{\partial^2 \lambda_{max}(x)}{(\partial x)^2}$ are known.

Harker's formula for the first derivative $\frac{\partial \lambda_{max}(x)}{\partial x}$

$$\left(\frac{\partial \lambda_{max}(A)}{\partial a_{ij}} \mid i > j \right) = \left([y(A)_i x(A)_j] - \frac{[y(A)_j x(A)_i]}{[a_{ij}]^2} \right)$$

where vectors $x(A), y(A)$ are the right-hand side and left-hand side eigenvectors of A , respectively.

Normalization $y(A)^T x(A) = 1$ is applied.

Harker's formula for the second derivative

$$\begin{aligned}
 \frac{\partial^2 \lambda_{max}(A)}{\partial_{ij} \partial_{kl}} = & (x(A)y(A)^T)_{li} Q_{jk}^+ + (x(A)y(A)^T)_{jk} Q_{li}^+ \\
 & \frac{(x(A)y(A)^T)_{ki} Q_{jl}^+ + (x(A)y(A)^T)_{jl} Q_{ki}^+}{[a_{kl}]^2} \\
 & \frac{(x(A)y(A)^T)_{lj} Q_{ik}^+ + (x(A)y(A)^T)_{ik} Q_{lj}^+}{[a_{ij}]^2} \\
 & + \frac{(x(A)y(A)^T)_{kl} Q_{il}^+ + (x(A)y(A)^T)_{il} Q_{kj}^+}{[a_{ij}]^2 [a_{kl}]^2}
 \end{aligned}$$

if $i \neq k$ or $j \neq l$,

Harker's formula for the second derivative

$$\begin{aligned} \frac{\partial^2 \lambda_{max}(A)}{\partial_{ij} \partial_{kl}} = & \frac{2(x(A)y(A)^T)_{ij}}{[a_{ij}]^3} + 2(x(A)y(A)^T)_{ji} Q_{ii}^+ \\ & - 2 \frac{(x(A)y(A)^T)_{ii} Q_{jj}^+ + (x(A)y(A)^T)_{jj} Q_{ii}^+}{[a_{ij}]^2} \\ & + 2 \frac{(x(A)y(A)^T)_{ij} Q_{ij}^+}{[a_{ij}]^4} \end{aligned}$$

if $i = k$ and $j = l$, where $Q = \lambda_{max}(A)I - A$ and Q^+ denotes the Moore-Penrose pseudoinverse of Q .

Multivariate Newton's method

$$\underline{t}_{n+1} = \underline{t}_n - [HL(\underline{t}_n)]^{-1} \nabla L(\underline{t}_n).$$

Applications of the results:

- A generalization of the Eigenvector Method for the incomplete case
- CR -inconsistency can be computed during the filling in process, as soon as a connected graph is given
- User may get an automatic warning in case of misprints, detected as a high jump in CR -inconsistency.

Questions:

- How many comparisons are needed, if less than $n(n - 1)/2$?
- Thresholds for warning the user?
- Other inconsistency indices, to be presented in the next talk.

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Thank you for attention.

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