Pairwise comparison matrices: an empirical research

Sándor Bozóki[1,2], **Linda Dezső**[3,4],
Attila Poesz[2], **József Temesi**[2]

[1] Computer and Automation Research Institute, Hungarian Academy of Sciences (MTA SZTAKI);

[2] Department of Operations Research and Actuarial Sciences Corvinus University of Budapest

[3] University of Szeged

[4] International Study Programs, Corvinus University of Budapest

17 June, 2011

Outline

- Empirical pairwise comparison matrices
- Experiments
- Results
- Questions

Incomplete pairwise comparison matrix
(= pairwise comparison matrix with missing elements)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & * & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & * \\ * & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & * & 1/a_{3n} & \dots & 1 \end{pmatrix}.$$

- Size of matrix:
  - 4x4
  - 6x6
  - 8x8
- Filling in order:
  - sequential
  - random
  - Ross
- Type of the problem:
  - subjective (summer houses),
  - objective (map)

Filling in order:

- sequential
- random
- Ross (balanced)

question / order	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.
sequential	A-B	A-C	A-D	A-E	A-F	B-C	B-D	B-E	B-F	C-D	C-E	C-F	D-E	D-F	E-F
random	A-F	B-E	A-C	F-E	C-D	B-D	B-F	A-E	C-E	A-D	E-D	C-F	B-C	A-D	B-A
Ross	A-B	F-D	E-A	C-B	E-F	A-C	B-D	F-A	D-C	E-B	A-D	C-E	B-F	D-E	C-F

Summer houses

A

B

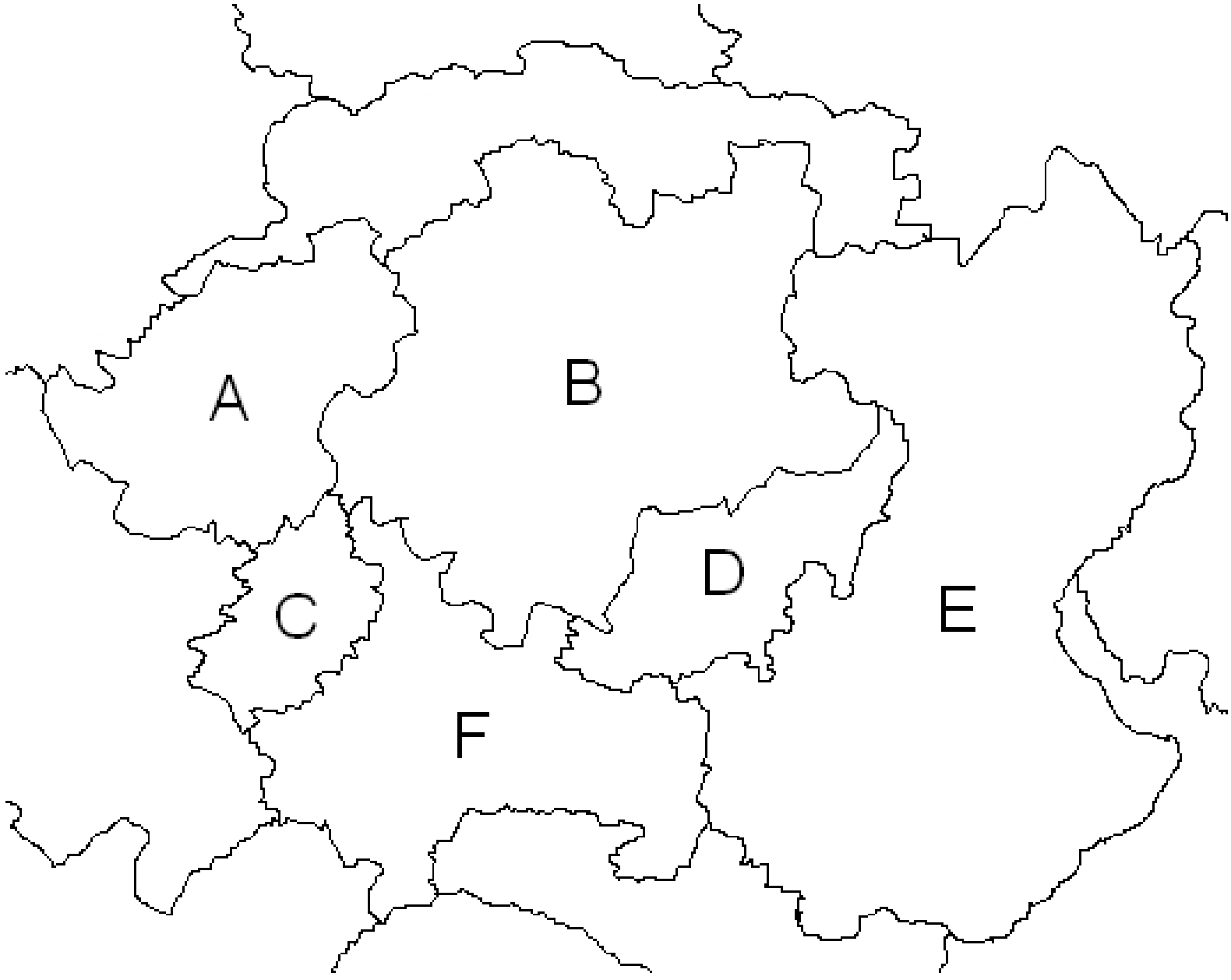
C

D

E

F

Map



Decision makers

270 business and economics students have been involved in the test exercises. We had groups of 22-26 students in each experiment.

Questionnaire

Special booklets have been designed for keeping the questioning orders. Turning back pages was not allowed.

Saaty defined the inconsistency ratio as $CR = \frac{\frac{\lambda_{max}-n}{n-1}}{RI_n}$, where λ_{max} is the Perron eigenvalue of the complete pairwise comparison matrix given by the decision maker, and RI_n is defined as $\frac{\overline{\lambda_{max}}-n}{n-1}$, where $\overline{\lambda_{max}}$ is an average value of the Perron eigenvalues of randomly generated $n \times n$ pairwise comparison matrices.

It is well known that $\lambda_{max} \geq n$ and equals to n if and only if the matrix is consistent, i.e., the transitivity property holds. It follows from the definition that CR is a positive linear transformation of λ_{max}.

According to Saaty, larger value of CR indicates higher level of inconsistency and the 10%-rule $(CR \leq 0.10)$ separates acceptable matrices from unacceptable ones.

Koczkodaj's CM index of inconsistency

$$\begin{pmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{pmatrix}$$

Koczkodaj defined the inconsistency index as

$$CM(a,b,c) = \min\left\{ \frac{1}{a}\left|a - \frac{b}{c}\right|, \frac{1}{b}\left|b - ac\right|, \frac{1}{c}\left|c - \frac{b}{a}\right| \right\}.$$

Koczkodaj's CM index of inconsistency

$$CM(a,b,c) = \min\left\{\frac{1}{a}\left|a-\frac{b}{c}\right|, \frac{1}{b}\left|b-ac\right|, \frac{1}{c}\left|c-\frac{b}{a}\right|\right\}.$$

Duszak and Koczkodaj extended this definition for a general $n \times n$ reciprocal matrix $\mathbf{A}$ as the maximum of $CM(a,b,c)$ for all triads (a,b,c), i.e., 3×3 submatrices which are themselves pairwise comparison matrices, in $\mathbf{A}$:

$$CM(\mathbf{A}) = \max\{CM(a_{ij}, a_{ik}, a_{jk}) \mid 1 \leq i < j < k \leq n\}.$$

Research questions

- Q1. Are inconsistency indices systematically higher in case of subjective type of problems?
- Q2. Are inconsistency indices higher in case of large size PC matrices?
- Q3. Has the questioning method an impact on the inconsistency?
- Q4. Is the behavior of the decision maker consequent in the course of the whole questioning procedure?
- Q5. What can we say about inconsistency and the weight vector if both are computed from incomplete data?

Results: average inconsistency levels of complete matrices

The average of CR and CM inconsistencies (in %) in case of complete matrices

	CR					
size / order	4x4		6x6		8x8	
	houses	maps	houses	maps	houses	maps
sequential	8,10	0,67	10,75	0,81	12,46	1,31
random	10,38	0,78	9,47	0,86	13,10	2,51
Ross	8,75	0,70	10,63	0,94	13,31	1,73
together	9,06	0,71	10,28	0,87	12,96	1,86

CM					
4x4		6x6		8x8	
houses	maps	Houses	Maps	houses	maps
0,62	0,29	0,79	0,45	0,87	0,54
0,68	0,31	0,77	0,46	0,86	0,57
0,60	0,28	0,82	0,46	0,90	0,58
0,63	0,29	0,79	0,46	0,88	0,56

Q1. Are inconsistency indices systematically higher in case of subjective type of problems?

A1. *Yes, definitely.*

Results: average inconsistency levels of complete matrices

The average of CR and CM inconsistencies (in %) in case of complete matrices

	CR					
size / order	4x4		6x6		8x8	
	houses	maps	houses	maps	houses	maps
sequential	8,10	0,67	10,75	0,81	12,46	1,31
random	10,38	0,78	9,47	0,86	13,10	2,51
Ross	8,75	0,70	10,63	0,94	13,31	1,73
together	9,06	0,71	10,28	0,87	12,96	1,86

CM					
4x4		6x6		8x8	
houses	maps	Houses	Maps	houses	maps
0,62	0,29	0,79	0,45	0,87	0,54
0,68	0,31	0,77	0,46	0,86	0,57
0,60	0,28	0,82	0,46	0,90	0,58
0,63	0,29	0,79	0,46	0,88	0,56

Q2. Are inconsistency indices higher in case of large size pairwise comparison matrices?

A2. *Yes.*

Results: average inconsistency levels of complete matrices

The average of CR and CM inconsistencies (in %) in case of complete matrices

	CR					
size / order	4x4		6x6		8x8	
	houses	maps	houses	maps	houses	maps
sequential	8,10	0,67	10,75	0,81	12,46	1,31
random	10,38	0,78	9,47	0,86	13,10	2,51
Ross	8,75	0,70	10,63	0,94	13,31	1,73
together	9,06	0,71	10,28	0,87	12,96	1,86

CM					
4x4		6x6		8x8	
houses	maps	Houses	Maps	houses	maps
0,62	0,29	0,79	0,45	0,87	0,54
0,68	0,31	0,77	0,46	0,86	0,57
0,60	0,28	0,82	0,46	0,90	0,58
0,63	0,29	0,79	0,46	0,88	0,56

Q3. Has the questioning method an impact on the inconsistency?

A3. *No.*

Results: average inconsistency levels through the filling in process

Table 3. The average of CR inconsistencies (in %) in case of 6×6 incomplete matrices

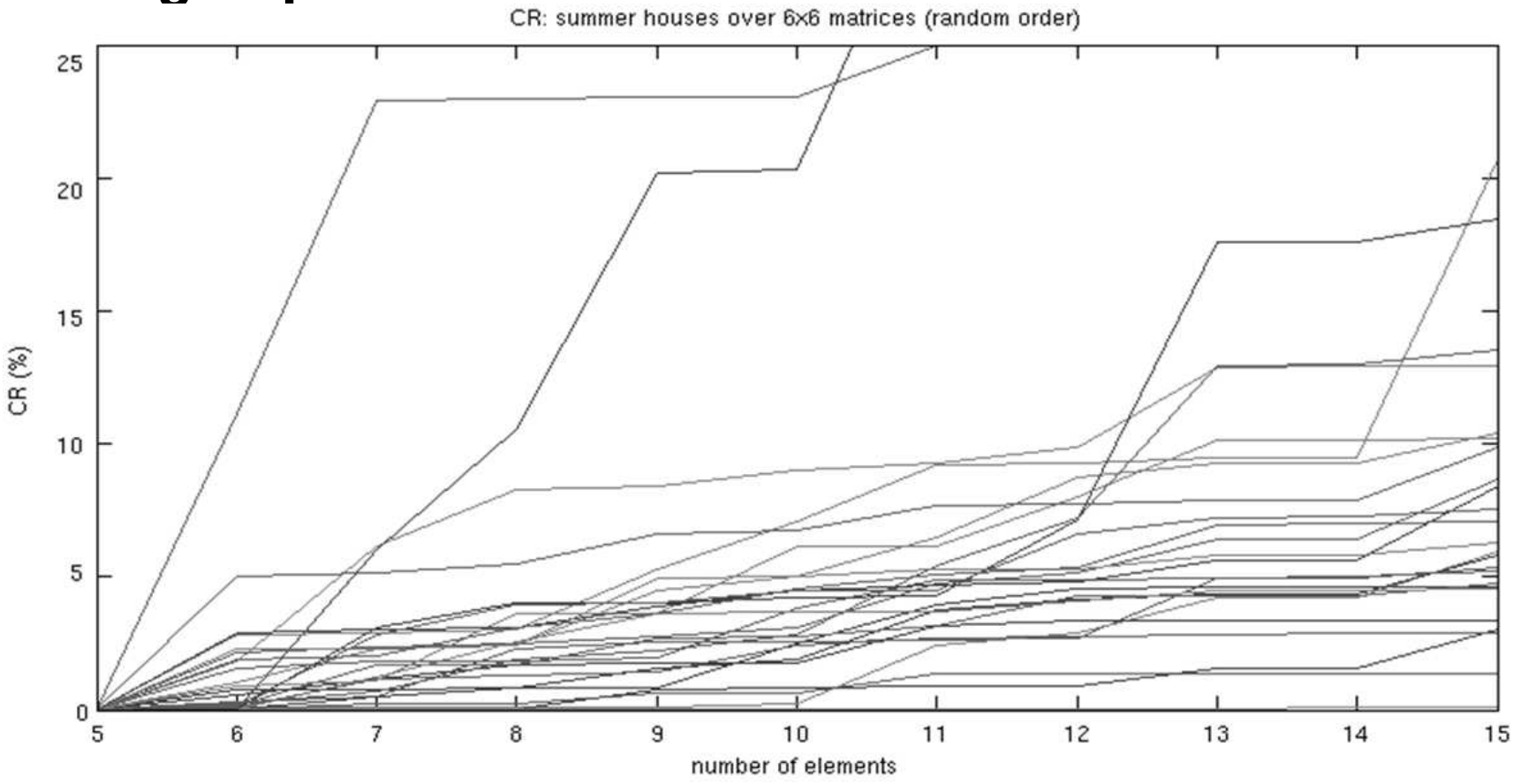
Table 3a. Summer houses

number of matrix elements / Order	5	6	7	8	9	10	11	12	13	14	15
sequential	0,00	0,96	1,82	3,71	4,74	5,66	6,61	7,33	8,35	9,21	10,75
random	0,01	1,38	2,77	3,49	4,42	4,97	6,25	6,91	8,17	8,19	9,47
Ross	0,01	1,37	2,50	3,84	4,93	5,45	6,27	7,24	7,85	9,52	10,63

Table 3b. Maps

number of matrix elements / Order	5	6	7	8	9	10	11	12	13	14	15
sequential	0,00	0,13	0,18	0,25	0,32	0,40	0,48	0,55	0,64	0,72	0,81
random	0,01	0,06	0,11	0,21	0,40	0,51	0,58	0,66	0,72	0,73	0,86
Ross	0,00	0,07	0,14	0,23	0,31	0,37	0,50	0,73	0,79	0,89	0,94

Results: individual inconsistency levels through the filling in process



Q4. Is the behavior of the decision maker consequent in the course of the whole questioning procedure?

A4. *Yes, for most of the decision makers.*

Results: weights from incomplete data

Table 4. Spearman rank correlation coefficients in case of 6×6 incomplete matrices

Type \ number of matrix elements	5	6	7	8	9	10	11	12	13	14	15
summer houses	0,82	0,88	0,90	0,92	0,93	0,94	0,96	0,97	0,97	0,98	1,00
map	0,97	0,97	0,98	0,98	0,98	0,98	0,99	0,99	0,99	1,00	1,00

Questions:

- How about real decisions?
- How to define rules for warning the decision maker when s/he is entering an element causing a high jump in inconsistency?

- Thank you for attention.

bozoki@sztaki.hu

http://www.sztaki.hu/∼bozoki