The logarithmic least squares optimality of the geometric mean of weight vectors calculated from all spanning trees for (in)complete pairwise comparison matrices

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incomplete pairwise comparison matrix

\[
A = \begin{pmatrix}
1 & a_{12} & a_{14} & a_{15} & a_{16} \\
 a_{21} & 1 & a_{23} & \\
 a_{32} & 1 & a_{34} & \\
 a_{41} & a_{43} & 1 & a_{45} & \\
 a_{51} & a_{54} & 1 & \\
 a_{61} & 1 & \\
\end{pmatrix}
\]
incomplete pairwise comparison matrix and its graph

\[
A = \begin{pmatrix}
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a_{21} & 1 & a_{23} & & \\
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a_{61} & & & & 1
\end{pmatrix}
\]
The Logarithmic Least Squares (LLS) problem

\[
\min \sum_{i, j : \text{ } a_{ij} \text{ is known}} \left[ \log a_{ij} - \log \left( \frac{w_i}{w_j} \right) \right]^2
\]

\[w_i > 0, \quad i = 1, 2, \ldots, n.\]

The most common normalizations are

\[\sum_{i=1}^{n} w_i = 1, \quad \prod_{i=1}^{n} w_i = 1\]

and \(w_1 = 1\).
**Theorem** (Bozóki, Fülöp, Rónyai, 2010): Let $A$ be an incomplete or complete pairwise comparison matrix such that its associated graph $G$ is connected. Then the optimal solution $w = \exp y$ of the logarithmic least squares problem is the unique solution of the following system of linear equations:

\[
(Ly)_i = \sum_{k : e(i,k) \in E(G)} \log a_{ik} \quad \text{for all } i = 1, 2, \ldots, n,
\]

\[
y_1 = 0
\]

where $L$ denotes the Laplacian matrix of $G$ ($\ell_{ii}$ is the degree of node $i$ and $\ell_{ij} = -1$ if nodes $i$ and $j$ are adjacent).
example

\[
\begin{pmatrix}
1 & a_{12} & a_{14} & a_{15} & a_{16} \\
a_{21} & 1 & a_{23} & & \\
a_{32} & 1 & a_{34} & & \\
a_{41} & a_{43} & 1 & a_{45} & \\
a_{51} & a_{54} & 1 & & \\
a_{61} & & & & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
4 & -1 & 0 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
-1 & 0 & -1 & 3 & -1 & 0 \\
-1 & 0 & 0 & -1 & 2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 (= 0) \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6
\end{pmatrix}
= \begin{pmatrix}
\log(a_{12} a_{14} a_{15} a_{16}) \\
\log(a_{21} a_{23}) \\
\log(a_{32} a_{34}) \\
\log(a_{41} a_{43} a_{45}) \\
\log(a_{51} a_{54}) \\
\log a_{61}
\end{pmatrix}
Pairwise Comparison Matrix Calculator (PCMC)

The logarithmic least squares optimal weight vector can be calculated at

pcmc.online

$CR$-minimal ($\lambda_{\text{max}}$-minimal) completion is also calculated.

PCMC deals with Pareto optimality (efficiency) of weight vectors, too.
Pareto optimality (efficiency)

Let \( A = [a_{ij}]_{i,j=1,...,n} \) be an \( n \times n \) pairwise comparison matrix and \( w = (w_1, w_2, \ldots, w_n)^\top \) be a positive weight vector.

**Definition:** weight vector \( w \) is called **efficient**, if there exists no positive weight vector \( w' = (w'_1, w'_2, \ldots, w'_n)^\top \) such that

\[
\left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \quad \text{for all } 1 \leq i, j \leq n,
\]

\[
\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \quad \text{for some } 1 \leq k, \ell \leq n.
\]

**Remark:** A weight vector \( w \) is efficient if and only if \( cw \) is efficient, where \( c > 0 \) is an arbitrary scalar.
\[
\begin{pmatrix}
1 & 1 & 4 & 9 \\
1 & 1 & 7 & 5 \\
1/4 & 1/7 & 1 & 4 \\
1/9 & 1/5 & 1/4 & 1
\end{pmatrix},
\quad \mathbf{w}^{EM} =
\begin{pmatrix}
0.404518 \\
0.436173 \\
0.110295 \\
0.049014
\end{pmatrix},
\]
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0.404518 \\
0.436173 \\
0.110295 \\
0.049014 \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
\mathbf{w}^{EM}_i/\mathbf{w}^{EM}_j
\end{pmatrix} =
\begin{pmatrix}
1 & 0.9274 & 3.6676 & 8.2531 \\
1.0783 & 1 & 3.9546 & 8.8989 \\
0.2727 & 0.2529 & 1 & 2.2503 \\
0.1212 & 0.1124 & 0.4444 & 1 \\
\end{pmatrix}
\]
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1 & 1 & 7 & 5 \\
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0.110295 \\
0.049014
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\]

\[
\begin{bmatrix}
\frac{w_i^{EM}}{w_j^{EM}}
\end{bmatrix} = \begin{pmatrix}
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1.0783 & 1 & 3.9546 & 8.8989 \\
0.2727 & 0.2529 & 1 & 2.2503 \\
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\]
Pareto optimality (efficiency)

See more in

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The spanning tree approach (Tsyganok, 2000, 2010)

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1 & a_{12} & a_{14} & a_{15} & a_{16} \\
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\end{pmatrix}
\]
The spanning tree approach

Every spanning tree induces a weight vector.

Natural ways of aggregation: arithmetic mean, geometric mean etc.
Theorem (Lundy, Siraj, Greco, 2017): The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete pairwise comparison matrices.
Theorem (Lundy, Siraj, Greco, 2017): The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete pairwise comparison matrices.

Theorem (Bozóki, Tsyganok): Let \( A \) be an incomplete or complete pairwise comparison matrix such that its associated graph is connected. Then the optimal solution of the logarithmic least squares problem is equal, up to a scalar multiplier, to the geometric mean of weight vectors calculated from all spanning trees.
proof

Let $G$ be the connected graph associated to the (in)complete pairwise comparison matrix $A$ and let $E(G)$ denote the set of edges. The edge between nodes $i$ and $j$ is denoted by $e(i, j)$.

The Laplacian matrix of graph $G$ is denoted by $L$. Let $T^1, T^2, \ldots, T^s, \ldots, T^S$ denote the spanning trees of $G$, where $S$ denotes the number of spanning trees. $E(T^s)$ denotes the set of edges in $T^s$.

Let $w^s, s = 1, 2, \ldots, S$, denote the weight vector calculated from spanning tree $T^s$. Weight vector $w^s$ is unique up to a scalar multiplication. Assume without loss of generality that $w_1^s = 1$.

Let $y^s := \log w^s, s = 1, 2, \ldots, S$, where the logarithm is taken element-wise.
proof

Let $w^{LLS}$ denote the optimal solution to the incomplete Logarithmic Least Squares problem (normalized by $w_1^{LLS} = 1$) and $y^{LLS} := \log w^{LLS}$, then

$$
\left( Ly^{LLS} \right)_i = \sum_{k : e(i,k) \in E(G)} b_{ik} \quad \text{for all } i = 1, 2, \ldots, n,
$$

where $b_{ik} = \log a_{ik}$ for all $e(i, k) \in E(G)$.

$b_{ik} = -b_{ki}$ for all $e(i, k) \in E(G)$.

In order to prove the theorem, it is sufficient to show that

$$
\left( L \frac{1}{S} \sum_{s=1}^{S} y^s \right)_i = \sum_{k : e(i,k) \in E(G)} b_{ik} \quad \text{for all } i = 1, 2, \ldots, n.
$$
proof

Challenge: the Laplacian matrices of the spanning trees are different from the Laplacian of $G$.

Consider an arbitrary spanning tree $T^s$. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i, j) \in E(T^s)$.

Introduce the incomplete pairwise comparison matrix $A^s$ by $a_{ij}^s := a_{ij}$ for all $e(i, j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i, j) \in E(G) \setminus E(T^s)$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.

Note that the Laplacian matrices of $A$ and $A^s$ are the same (L).
proof

\[
\begin{pmatrix}
1 & a_{12} & a_{14} & a_{15} & a_{16} \\
a_{21} & 1 & a_{23} & & \\
a_{32} & 1 & & & \\
a_{41} & & a_{32}a_{21}a_{14} & & \\
a_{51} & & & a_{41}a_{15} & \\
a_{61} & & & & 1 \\
\end{pmatrix}
\]
proof

Since weight vector $w^s$ is generated by the matrix elements belonging to spanning tree $T^s$, it is the optimal solution of the $LLS$ problem regarding $A^s$, too. Equivalently, the following system of linear equations holds.

$$(Ly^s)_i = \sum_{k:e(i,k) \in E(T^s)} b_{ik} + \sum_{k:e(i,k) \in E(G) \setminus E(T^s)} b^s_{ik} \text{ for all } i = 1, \ldots, n$$
proof

Lemma

\[ \sum_{s=1}^{S} \left( \sum_{k:e(i,k) \in E(T^s)} b_{ik} + \sum_{k:e(i,k) \in E(G) \setminus E(T^s)} b_{ik}^s \right) = S \sum_{k:e(i,k) \in E(G)} b_{ik} \]
proof of the lemma

\[ T^1 \]
proof of the lemma

\[ T^1 \]
proof of the lemma

$T^1$
proof of the lemma

\[ b^{1}_{12} = b_{15} + b_{54} + b_{43} + b_{32} \]
proof of the lemma

\[ b_{12}^1 = b_{15} + b_{54} + b_{43} + b_{32} \]
proof of the lemma

\[ b_{12}^1 = b_{15} + b_{54} + b_{43} + b_{32} \]

\[ b_{15}^4 = b_{12} + b_{23} + b_{34} + b_{45} \]
proof of the lemma

\[ b_{12}^1 = b_{15} + b_{54} + b_{43} + b_{32} \]

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proof of the lemma

\[ b_{12}^1 = b_{15} + b_{54} + b_{43} + b_{32} \]

\[ b_{15}^4 = b_{12} + b_{23} + b_{34} + b_{45} \]

\[ b_{12}^1 + b_{15}^4 = b_{12} + b_{15} \]
proof of the lemma

\[ b_{12}^1 = b_{15} + b_{54} + b_{43} + b_{32} \]

\[ b_{15}^4 = b_{12} + b_{23} + b_{34} + b_{45} \]

\[ b_{12}^1 + b_{15}^4 = b_{12} + b_{15} \]
proof of the lemma

\[ b^{1}_{12} = b_{15} + b_{54} + b_{43} + b_{32} \]

\[ b^{4}_{15} = b_{12} + b_{23} + b_{34} + b_{45} \]

\[ b^{1}_{12} + b^{4}_{15} = b_{12} + b_{15} \]
proof

Finally, to complete the proof, take the sum of equations

\[(Ly^s)_i = \sum_{k:e(i,k) \in E(T^s)} b_{ik} + \sum_{k:e(i,k) \in E(G) \setminus E(T^s)} b_{ik}^s \quad \text{for all } i = 1, \ldots, n\]

for all \( s = 1, 2, \ldots, S \) and apply the lemma

\[
\sum_{s=1}^{S} \left( \sum_{k:e(i,k) \in E(T^s)} b_{ik} + \sum_{k:e(i,k) \in E(G) \setminus E(T^s)} b_{ik}^s \right) = S \sum_{k:e(i,k) \in E(G)} b_{ik}
\]

to conclude that

\[y^{LLS} = \frac{1}{S} \sum_{s=1}^{S} y^s.\]
Remarks

Complete pairwise comparison matrices \((S = n^{n-2})\) are included in our theorem as a special case, and our proof can also be considered as a second, and shorter proof of the theorem of Lundy, Siraj and Greco (2017).

Special incomplete cases, investigated by Harker (1987); van Uden (2002); Chen, Kou, Tarn, Song (2015); Bozóki (2017) are also included.
Conclusions

The equivalence of two fundamental weighting methods has been shown.

The advantages of two approaches have been united.
Main references 1/4


Tsyganok, V. (2010): Investigation of the aggregation effectiveness of expert estimates obtained by the pairwise comparison method. Mathematical and Computer Modelling, 52(3-4) 538–544


Main references 2/4


Main references 3/4


Main references 4/4


Thank you for attention.

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