The logarithmic least squares optimality of the geometric mean of weight vectors calculated from all spanning trees for (in)complete pairwise comparison matrices

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IFORS, Québec City
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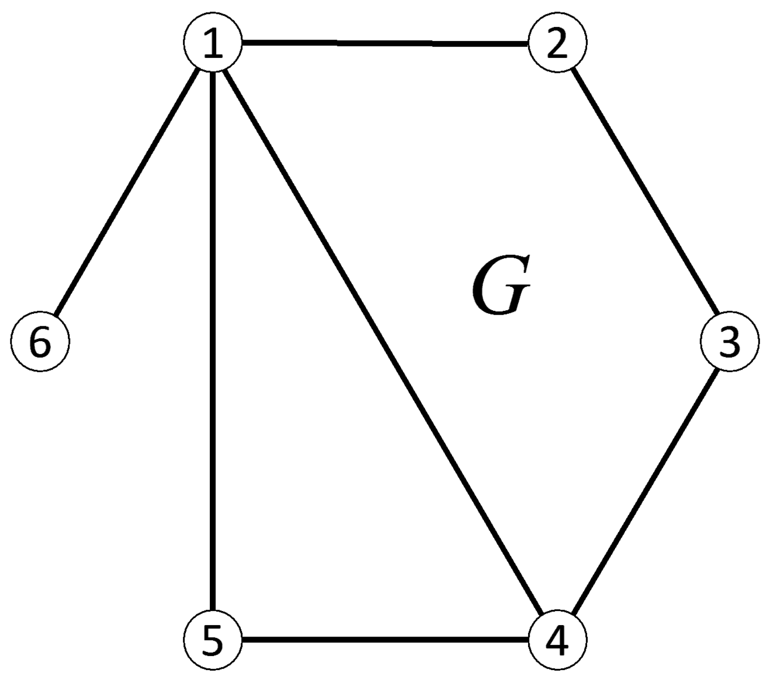
incomplete pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$

incomplete pairwise comparison matrix and its graph

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



The Logarithmic Least Squares (LLS) problem

$$\min \sum_{\substack{i,j: \\ a_{ij} \text{ is known}}} \left[\log a_{ij} - \log \left(\frac{w_i}{w_j} \right) \right]^2$$

$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

The most common normalizations are $\sum_{i=1}^{n} w_i = 1$, $\prod_{i=1}^{n} w_i = 1$ and $w_1 = 1$.

Theorem (Bozóki, Fülöp, Rónyai, 2010): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph G is connected. Then the optimal solution $\mathbf{w} = \exp \mathbf{y}$ of the logarithmic least squares problem is the unique solution of the following system of linear equations:

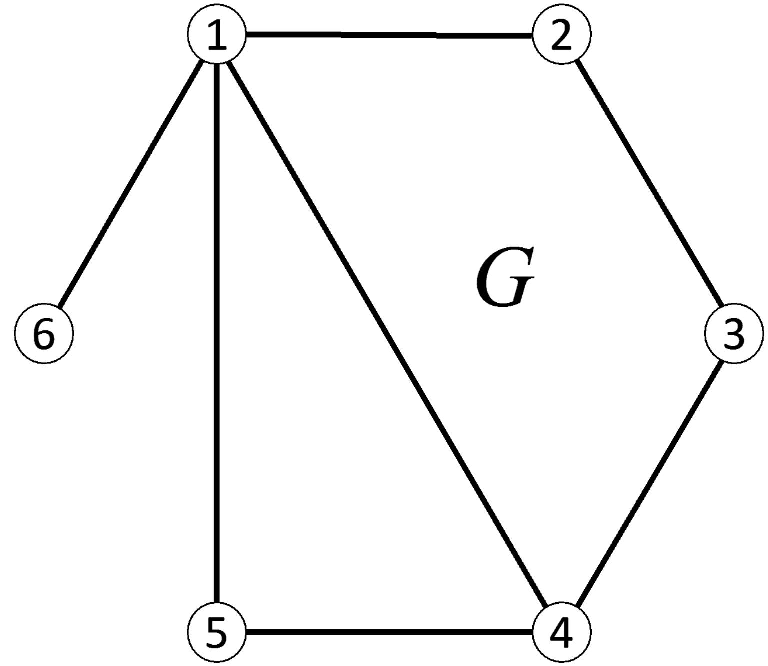
$$(\mathbf{Ly})_i = \sum_{k:e(i,k)\in E(G)} \log a_{ik} \qquad \text{for all } i = 1, 2, \ldots, n,$$

$$y_1 = 0$$

where $\mathbf{L}$ denotes the Laplacian matrix of G (ℓ_{ii} is the degree of node i and $\ell_{ij} = -1$ if nodes i and j are adjacent).

example

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 (= 0) \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} \log(a_{12}\, a_{14}\, a_{15}\, a_{16}) \\ \log(a_{21}\, a_{23}) \\ \log(a_{32}\, a_{34}) \\ \log(a_{41}\, a_{43}\, a_{45}) \\ \log(a_{51}\, a_{54}) \\ \log a_{61} \end{pmatrix}$$

Pairwise Comparison Matrix Calculator (PCMC)

The logarithmic least squares optimal weight vector can be calculated at

pcmc.online

CR-minimal ($\lambda_{\max}$-minimal) completion is also calculated.

PCMC deals with Pareto optimality (efficiency) of weight vectors, too.

Pareto optimality (efficiency)

Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$ be an $n \times n$ pairwise comparison matrix and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector.

Definition: weight vector $\mathbf{w}$ is called *efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \le \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \le i, j \le n,$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \le k, \ell \le n.$$

Remark: A weight vector $\mathbf{w}$ is efficient if and only if $c\mathbf{w}$ is efficient , where $c > 0$ is an arbitrary scalar.

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix},$$

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$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

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\left[\frac{w_i'}{w_j'}\right] = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{3.9546} & \mathbf{8.8989} \\ \mathbf{1} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2529} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1124} & 0.1124 & 0.4444 & 1 \end{pmatrix}.
$$

Pareto optimality (efficiency)

See more in

Bozóki, S., Fülöp, J. (2017): Efficient weight vectors from pairwise comparison matrices, European Journal of Operational Research (in print)
DOI 10.1016/j.ejor.2017.06.033

The spanning tree approach (Tsyganok, 2000, 2010)

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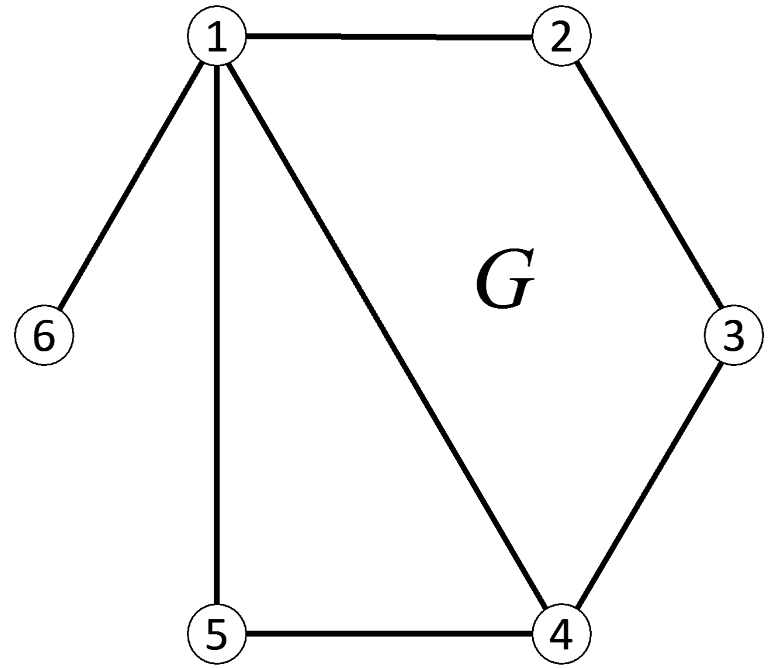

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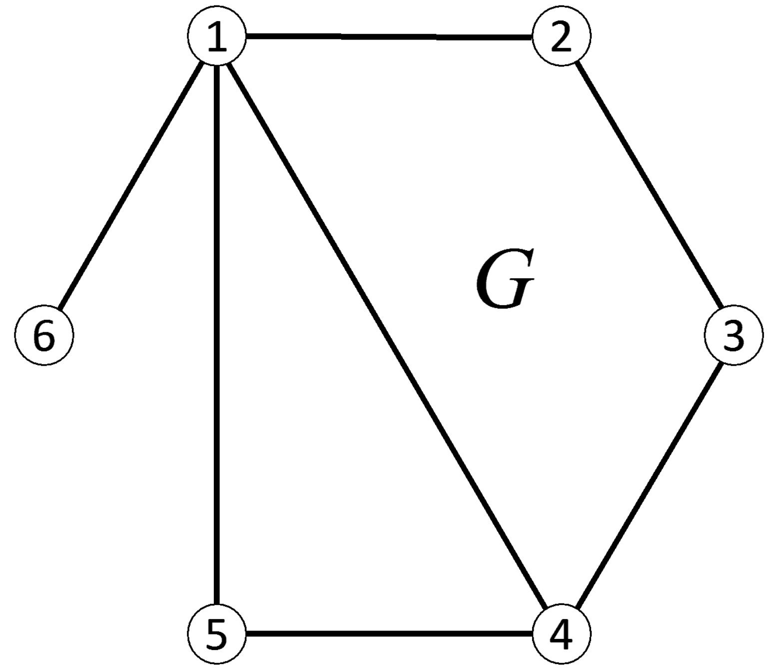

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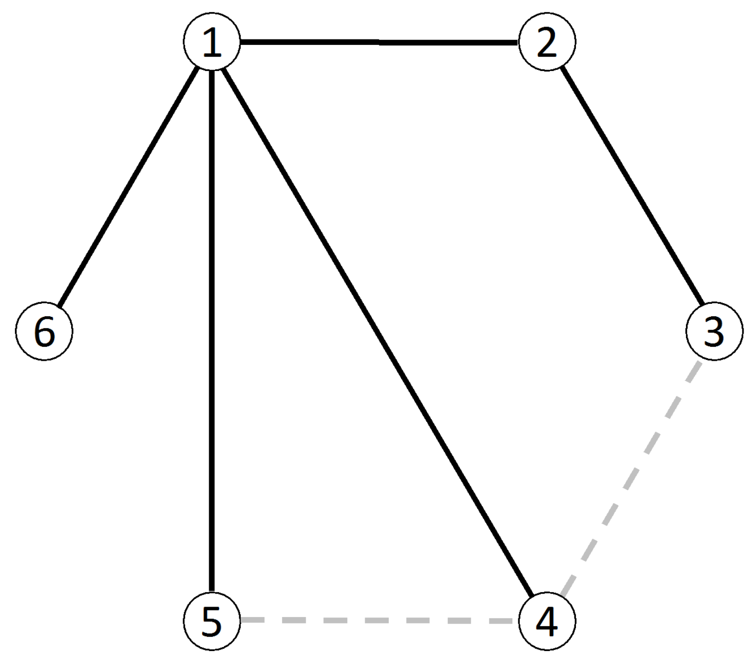

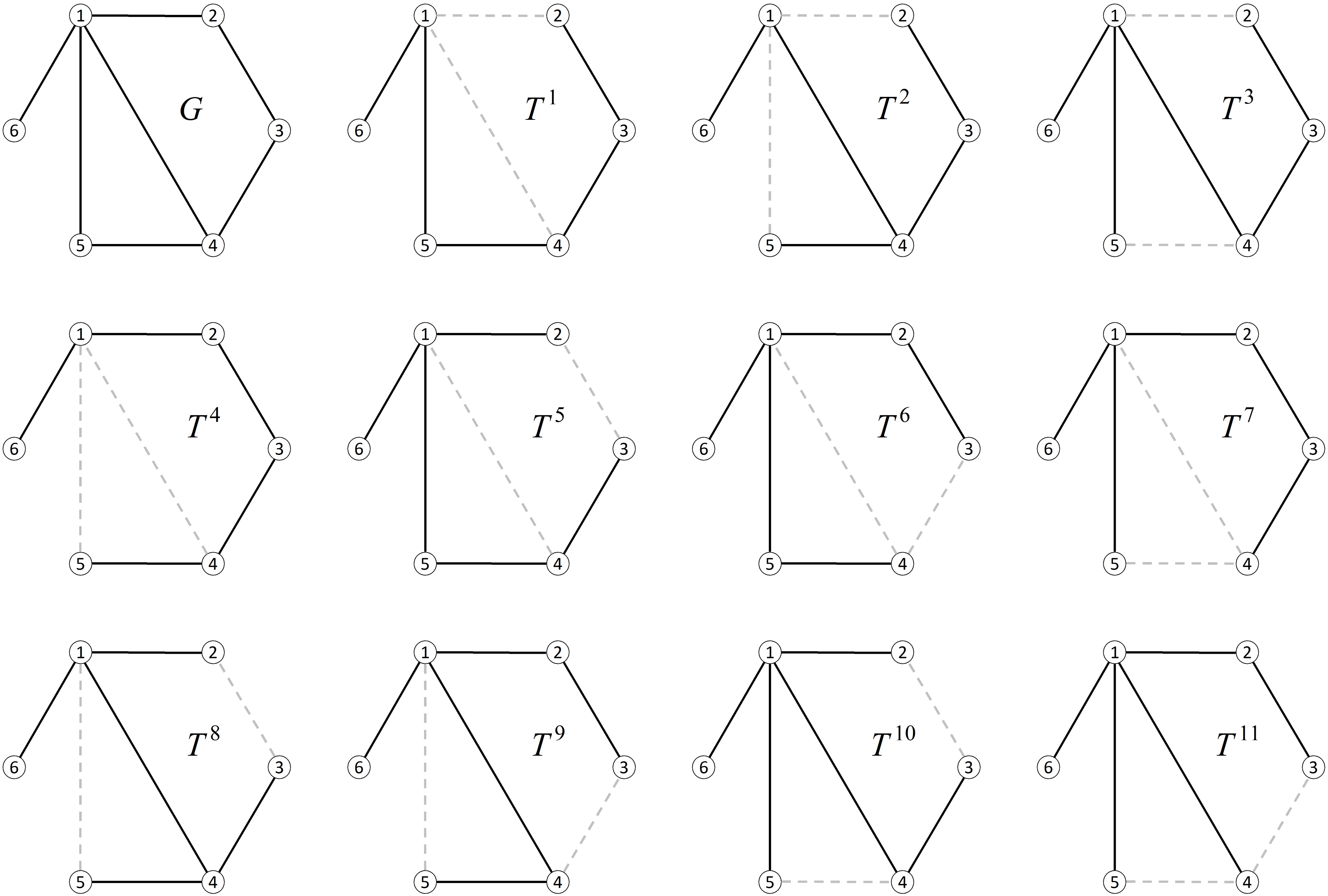
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The spanning tree approach

Every spanning tree induces a weight vector.

Natural ways of aggregation: arithmetic mean, geometric mean etc.

Theorem (Lundy, Siraj, Greco, 2017): The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete pairwise comparison matrices.

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Theorem (Bozóki, Tsyganok): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph is connected. Then the optimal solution of the logarithmic least squares problem is equal, up to a scalar multiplier, to the geometric mean of weight vectors calculated from all spanning trees.

proof

Let G be the connected graph associated to the (in)complete pairwise comparison matrix $\mathbf{A}$ and let $E(G)$ denote the set of edges. The edge between nodes i and j is denoted by $e(i, j)$.

The Laplacian matrix of graph G is denoted by $\mathbf{L}$. Let $T^1, T^2, \ldots, T^s, \ldots, T^S$ denote the spanning trees of G, where S denotes the number of spanning trees. $E(T^s)$ denotes the set of edges in T^s.

Let $\mathbf{w}^s, s = 1, 2, \ldots, S$, denote the weight vector calculated from spanning tree T^s. Weight vector $\mathbf{w}^s$ is unique up to a scalar multiplication. Assume without loss of generality that $w_1^s = 1$.

Let $\mathbf{y}^s := \log \mathbf{w}^s,\ s = 1, 2, \ldots, S$, where the logarithm is taken element-wise.

proof

Let $\mathbf{w}^{LLS}$ denote the optimal solution to the incomplete Logarithmic Least Squares problem (normalized by $w_1^{LLS} = 1$) and $\mathbf{y}^{LLS} := \log \mathbf{w}^{LLS}$, then

$$\left(\mathbf{L}\mathbf{y}^{LLS}\right)_i = \sum_{k: e(i,k) \in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \ldots, n,$$

where $b_{ik} = \log a_{ik}$ for all $e(i,k) \in E(G)$.

$b_{ik} = -b_{ki}$ for all $e(i,k) \in E(G)$.

In order to prove the theorem, it is sufficient to show that

$$\left(\mathbf{L}\frac{1}{S}\sum_{s=1}^{S}\mathbf{y}^s\right)_i = \sum_{k: e(i,k) \in E(G)} b_{ik} \qquad \text{for all } i = 1, 2, \ldots, n.$$

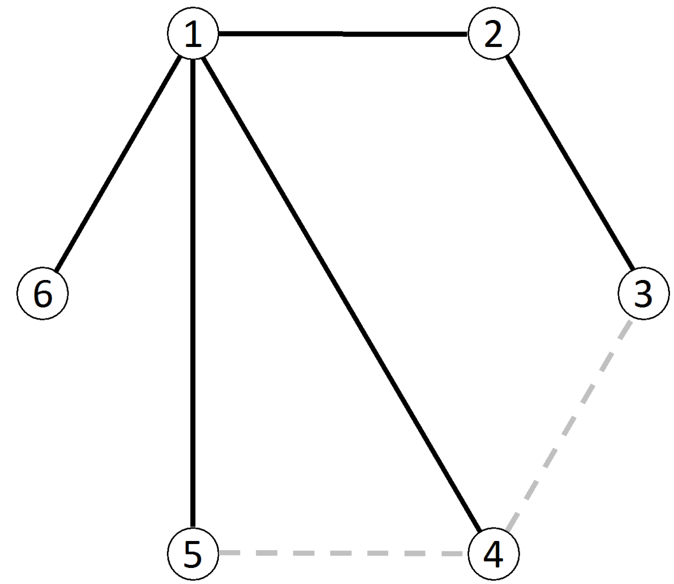
proof

Challenge: the Laplacian matrices of the spanning trees are different from the Laplacian of G.

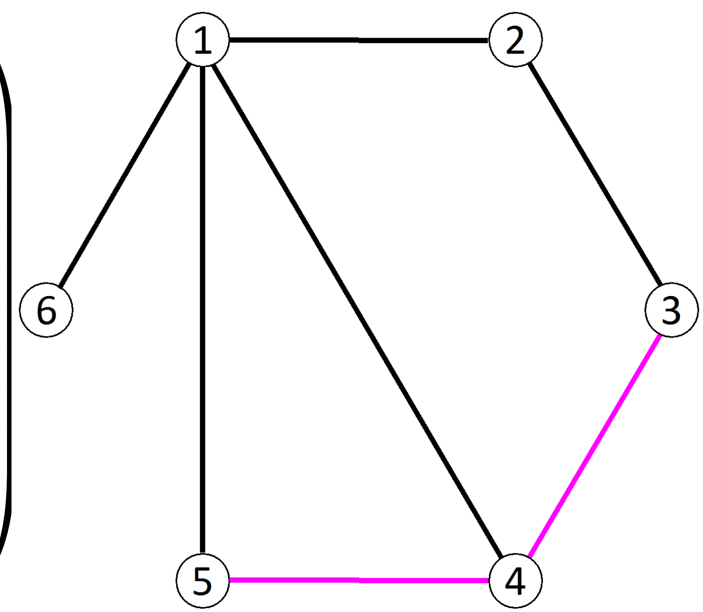
Consider an arbitrary spanning tree T^s. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$.
Introduce the incomplete pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G) \backslash E(T^s)$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.
Note that the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$).

proof

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & & & \\ a_{41} & & & 1 & & \\ a_{51} & & & & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{32}a_{21}a_{14} & & \\ a_{41} & & a_{41}a_{12}a_{23} & 1 & a_{41}a_{15} & \\ a_{51} & & & a_{51}a_{14} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



proof

Since weight vector $\mathbf{w}^s$ is generated by the matrix elements belonging to spanning tree T^s, it is the optimal solution of the LLS problem regarding $\mathbf{A}^s$, too. Equivalently, the following system of linear equations holds.

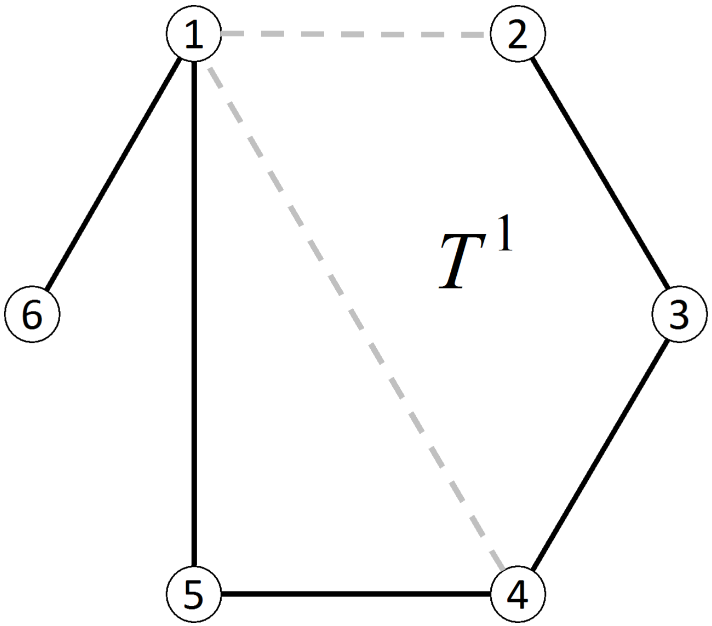
$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b^s_{ik} \quad \text{for all } i = 1, \ldots, n$$

proof

Lemma

$$\sum_{s=1}^{S}\left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^{s}\right) = S\sum_{k:e(i,k)\in E(G)} b_{ik}$$

proof of the lemma



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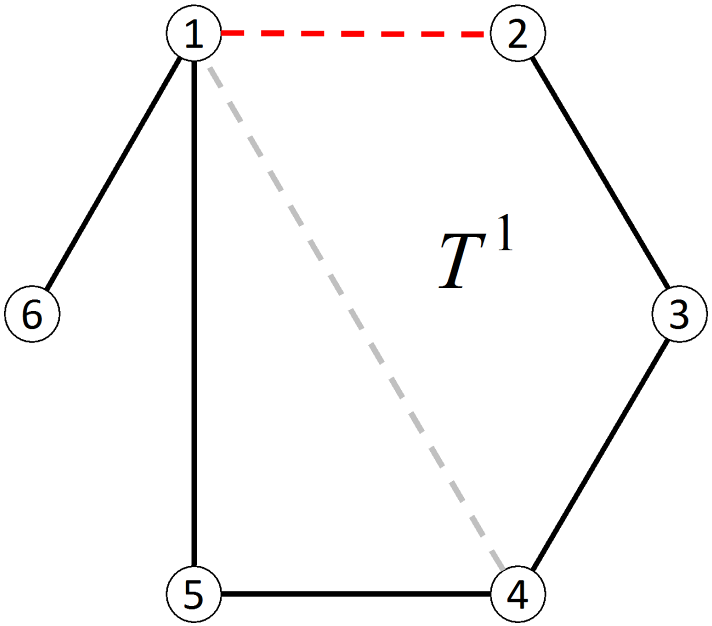

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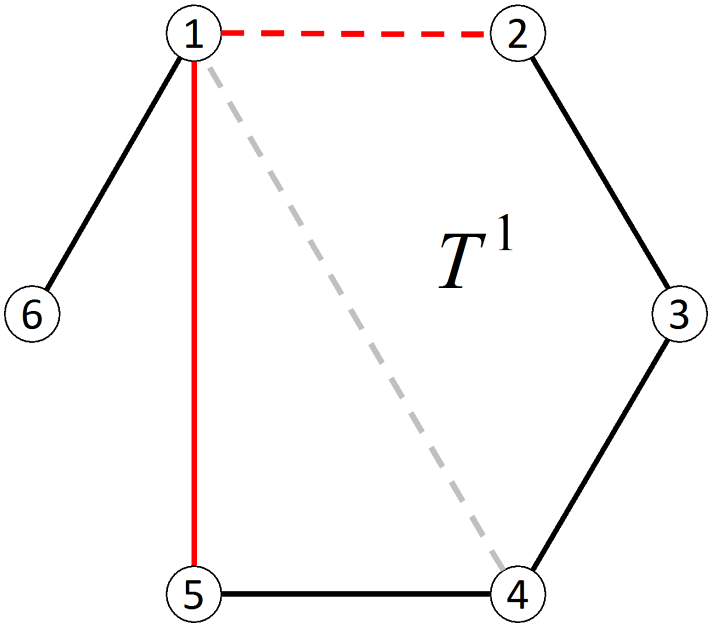

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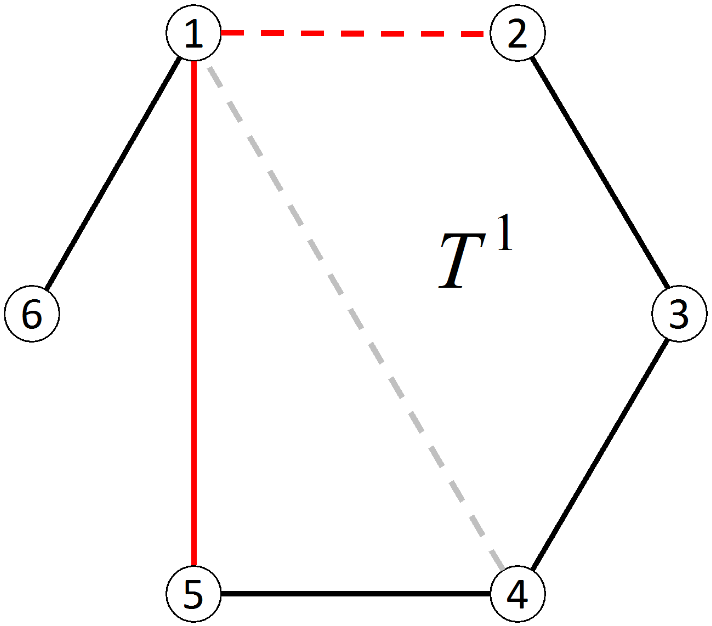
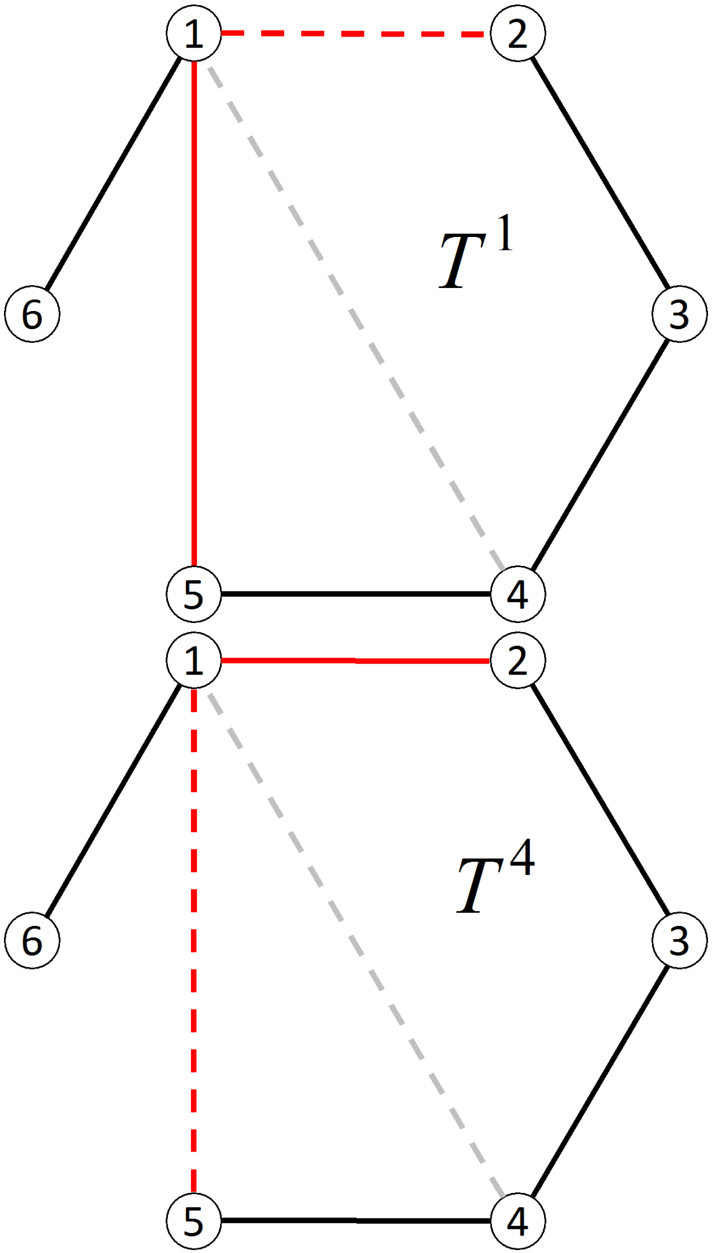


$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

proof of the lemma



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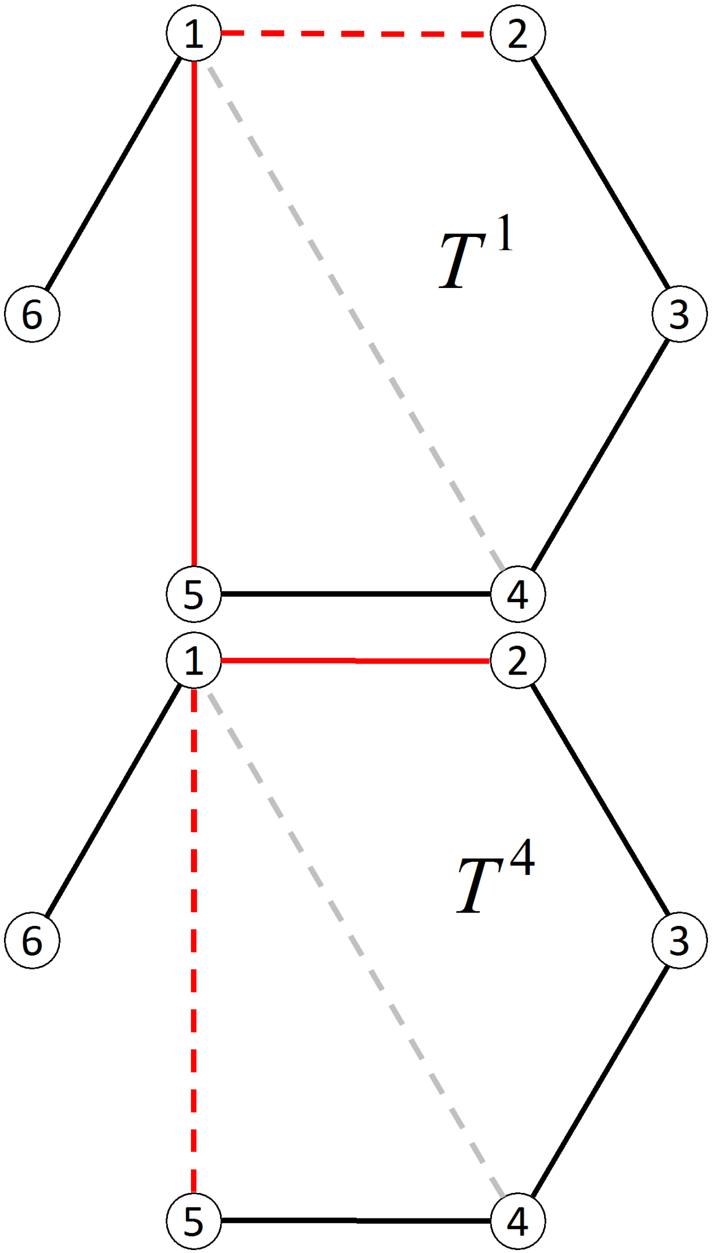

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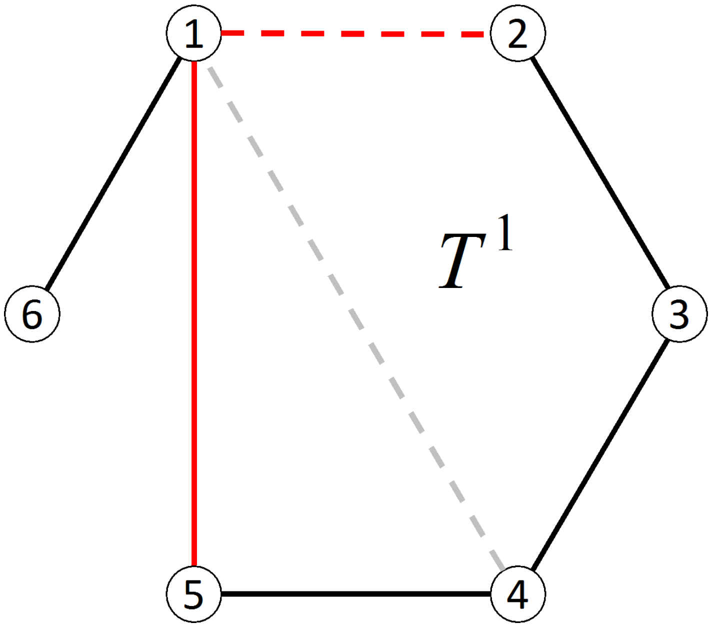

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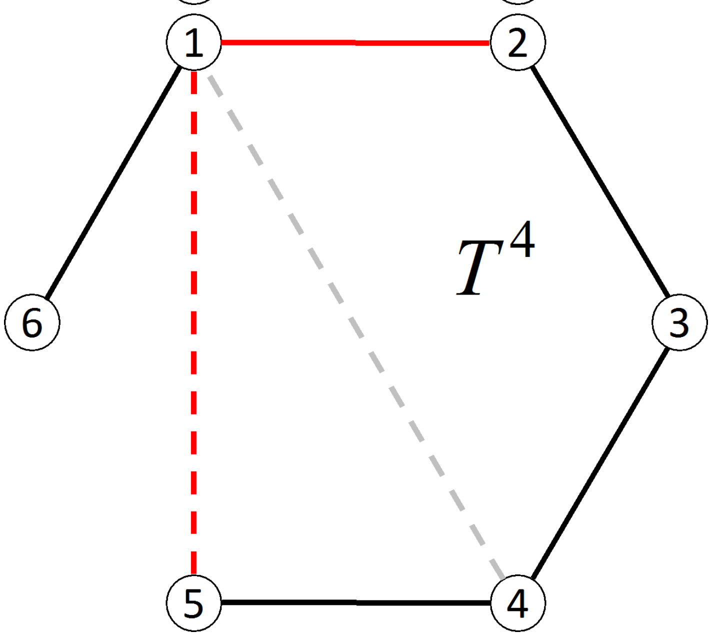

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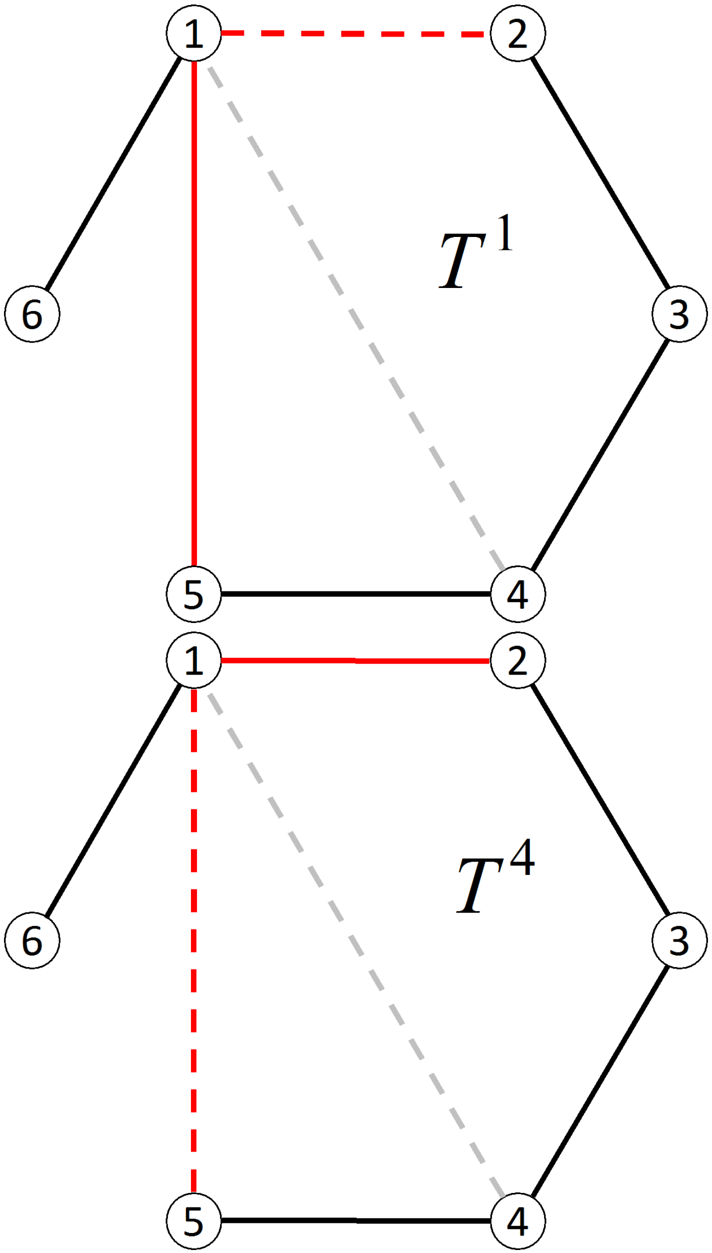

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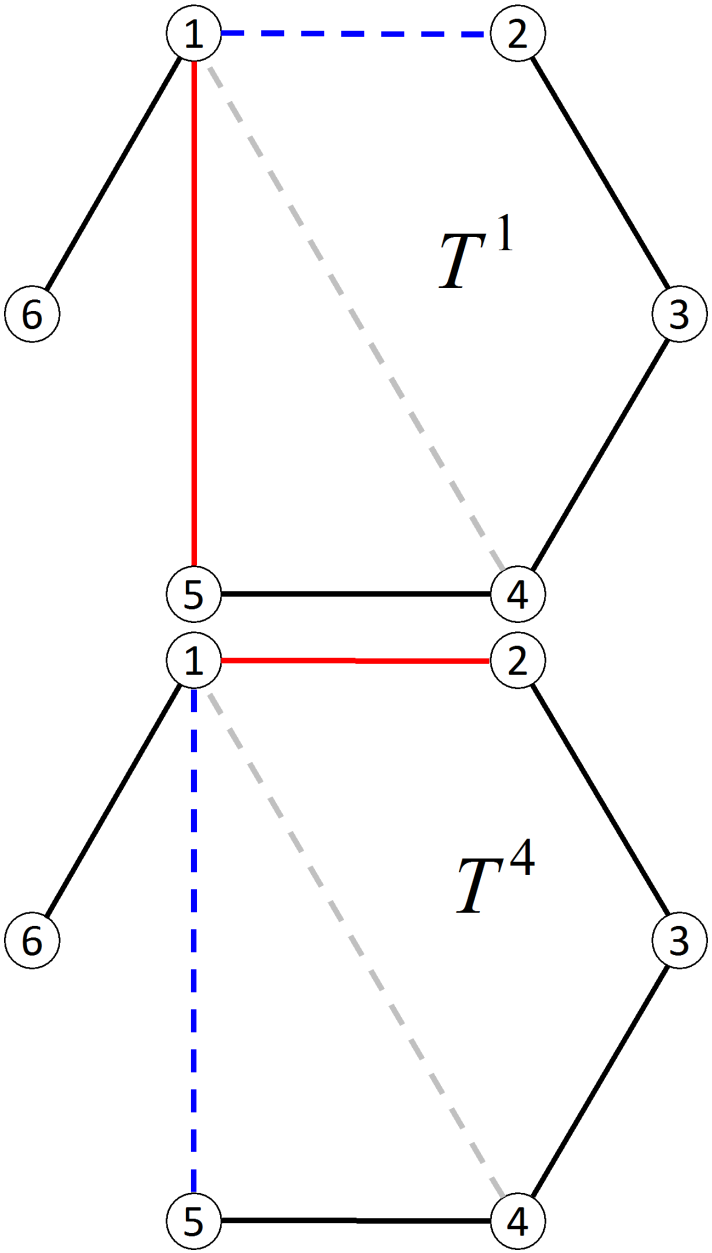

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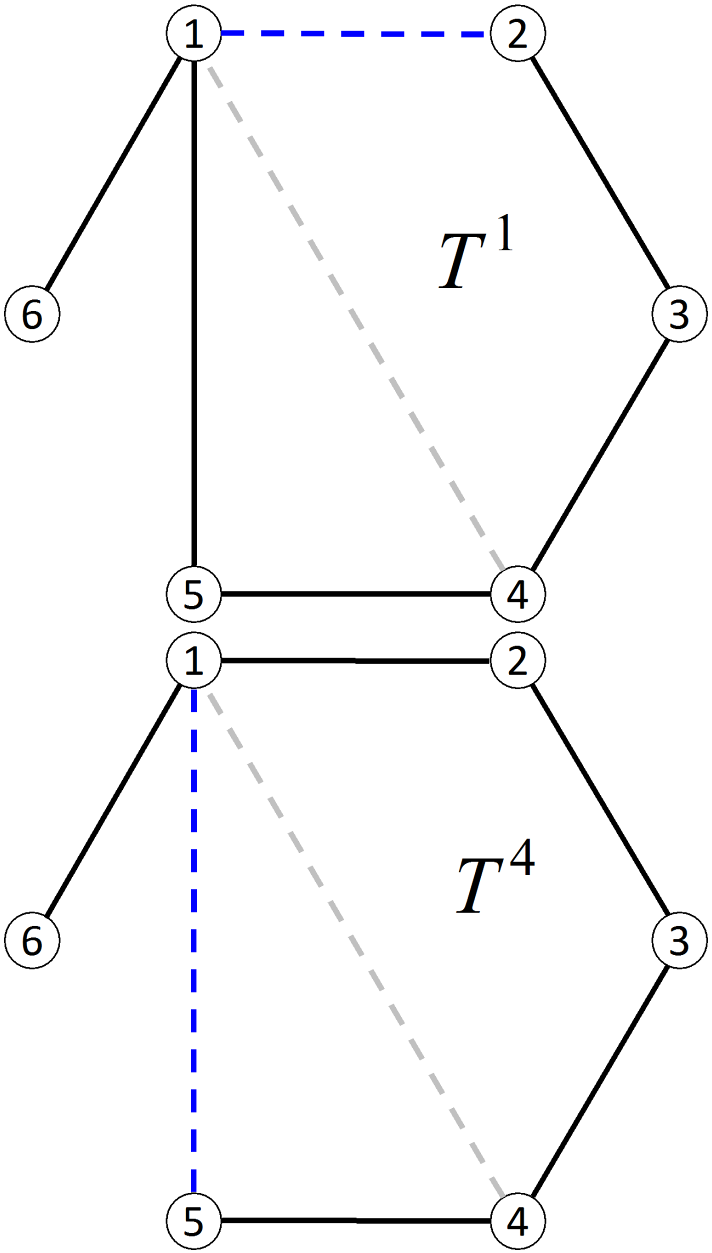

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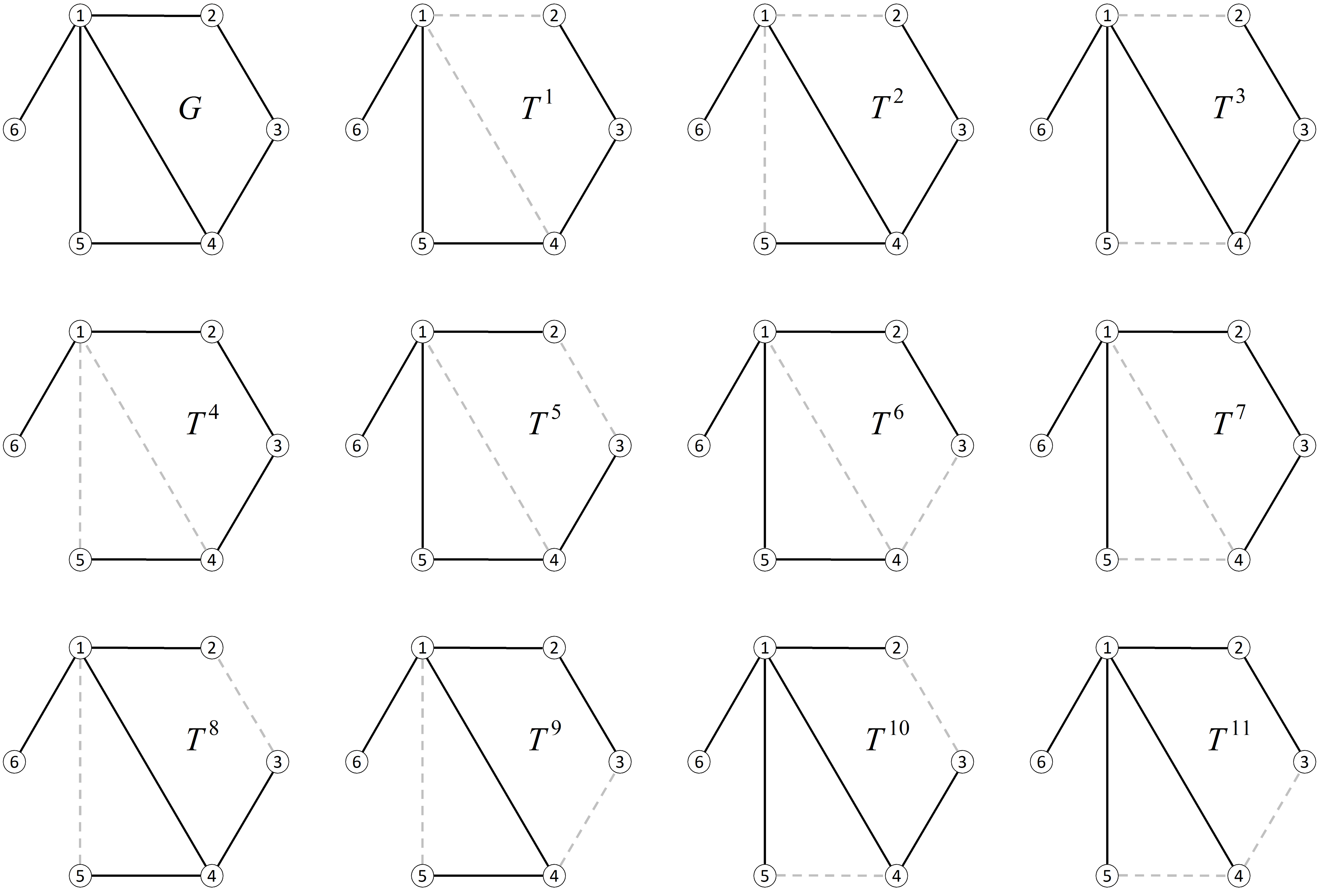

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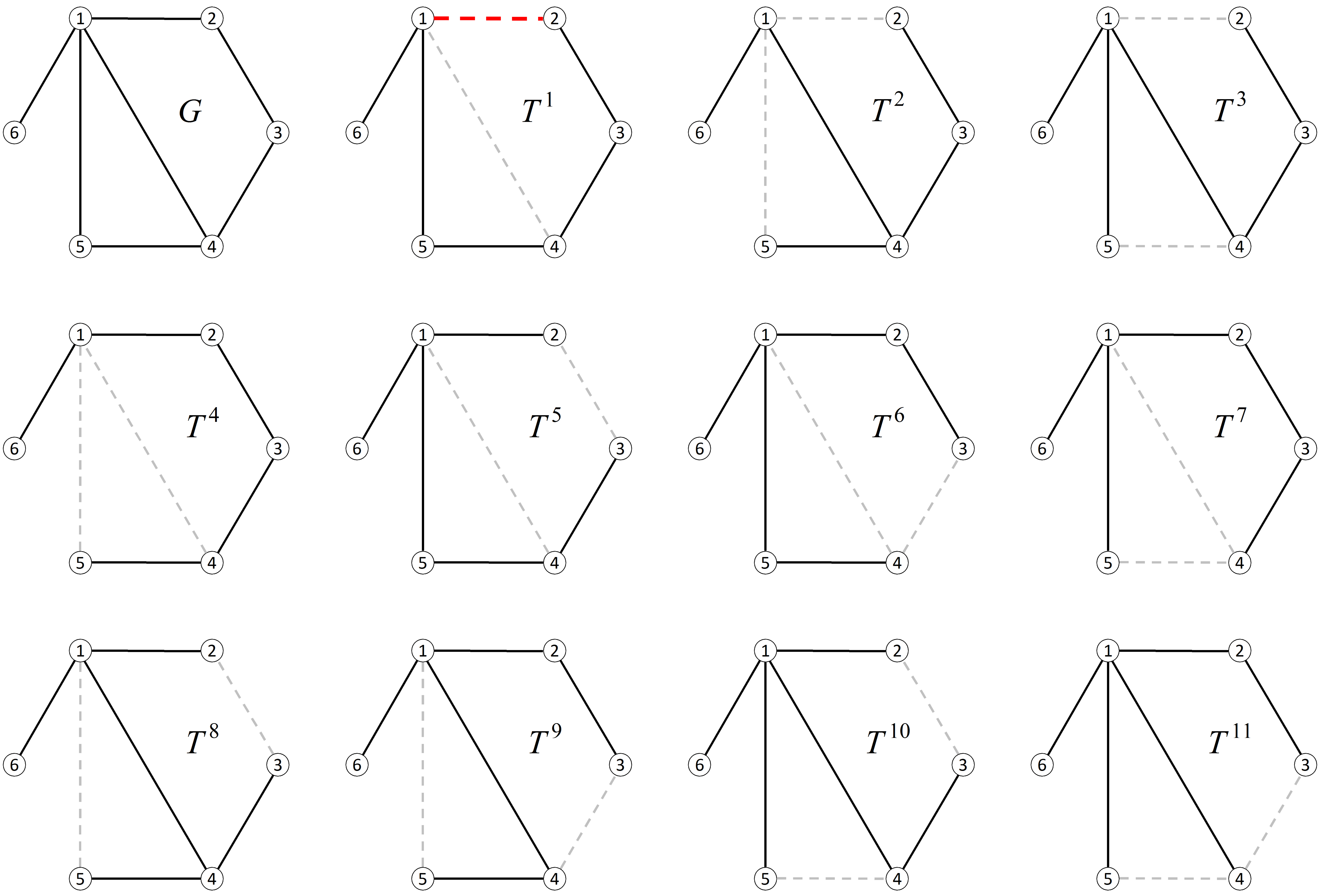

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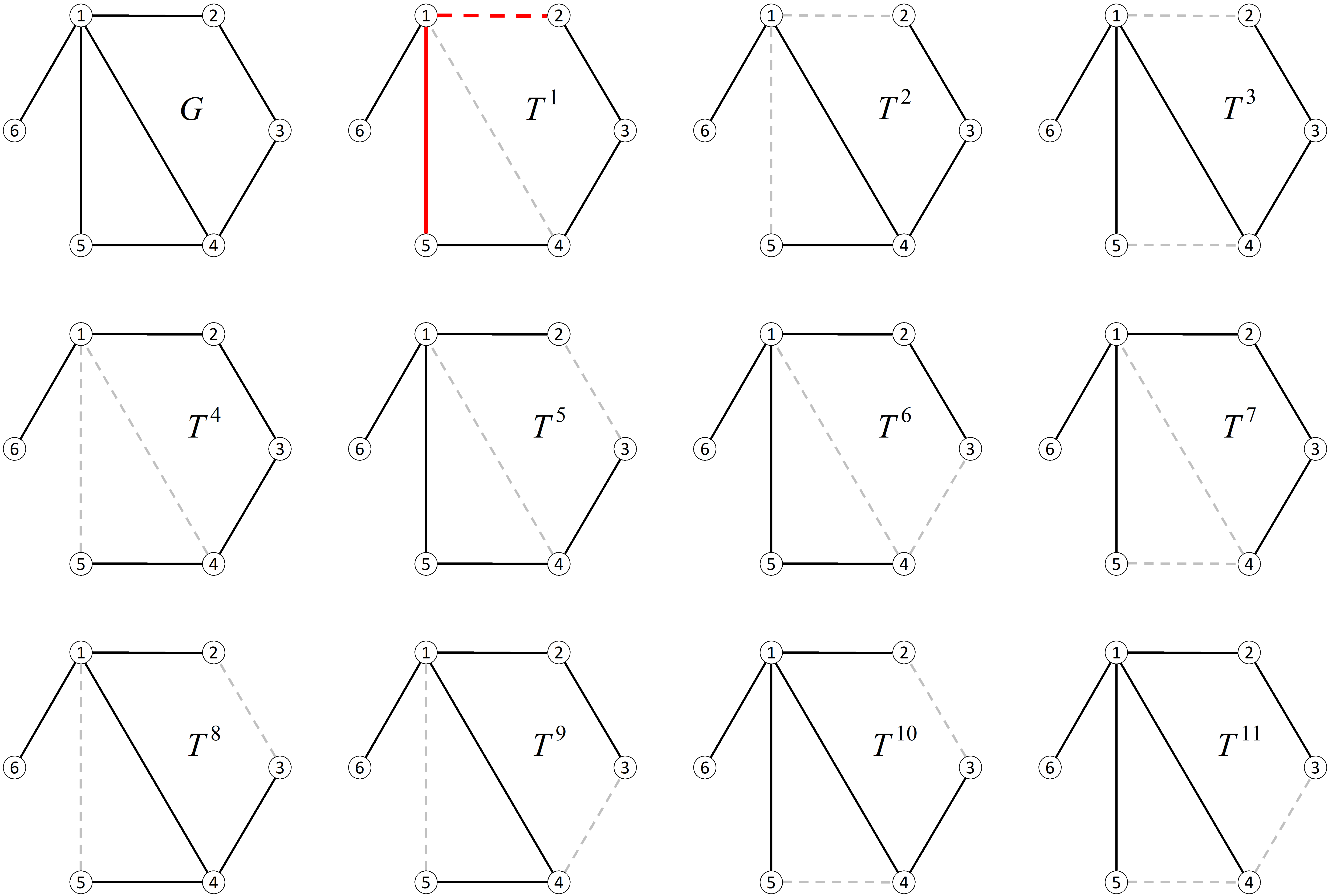
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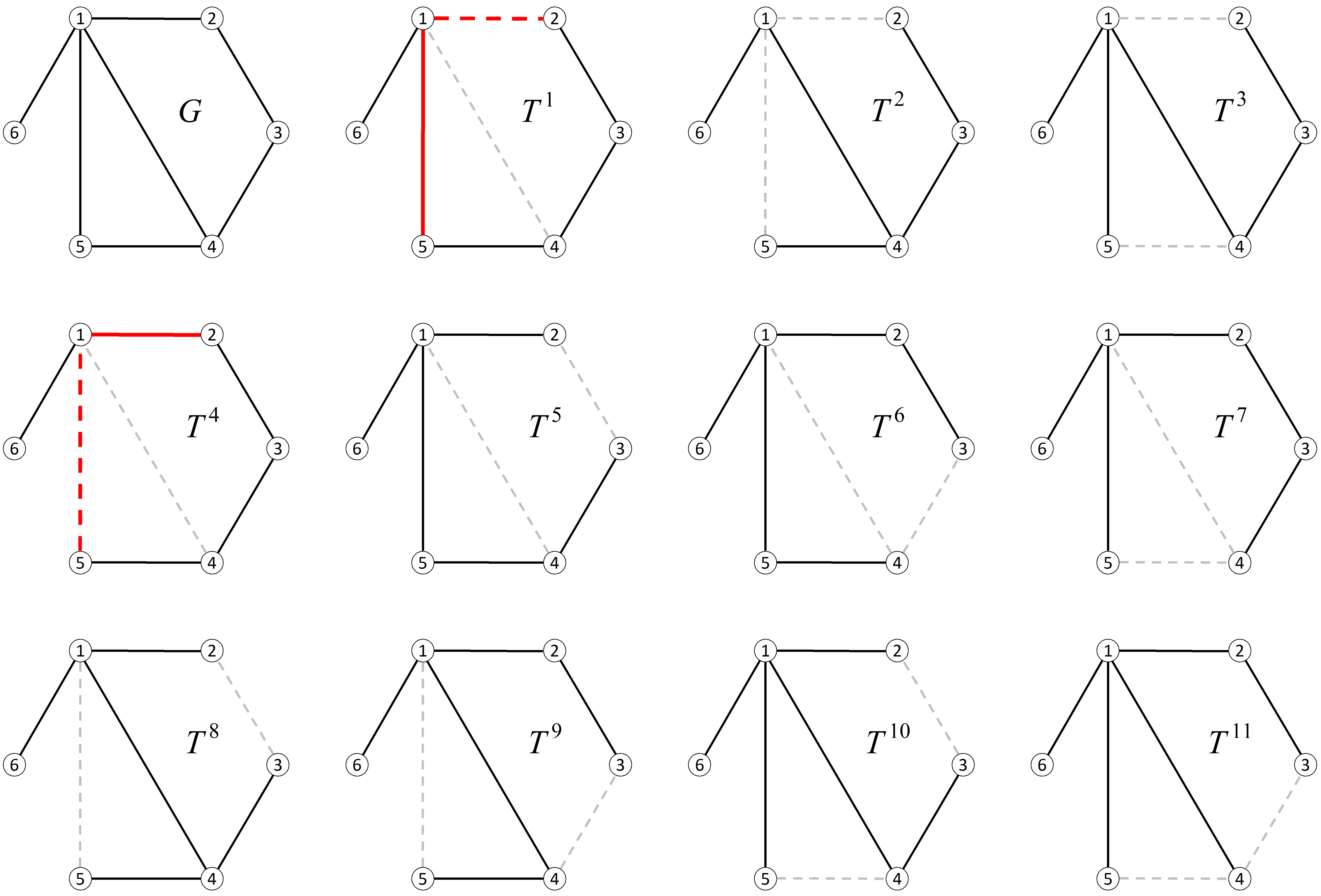

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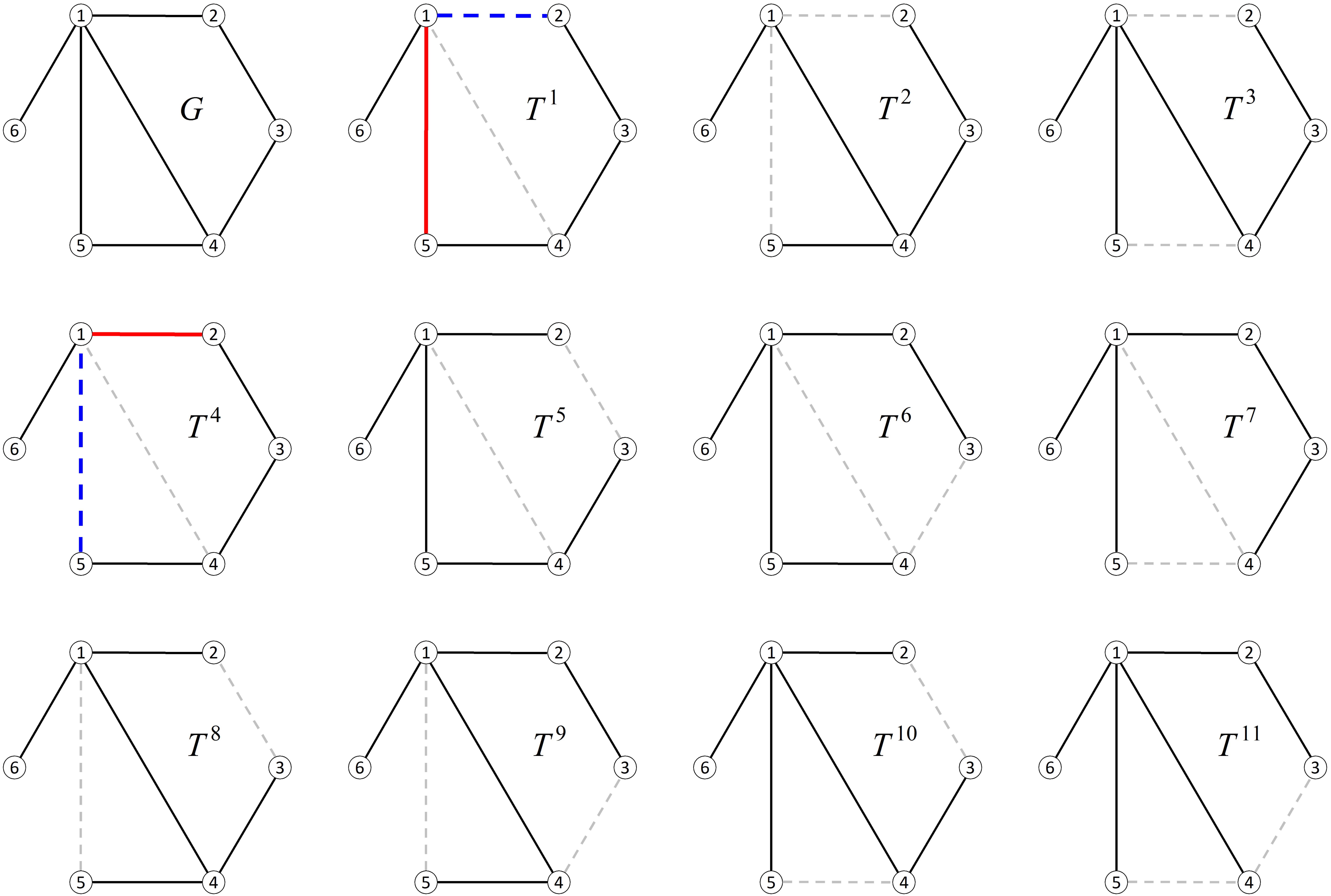
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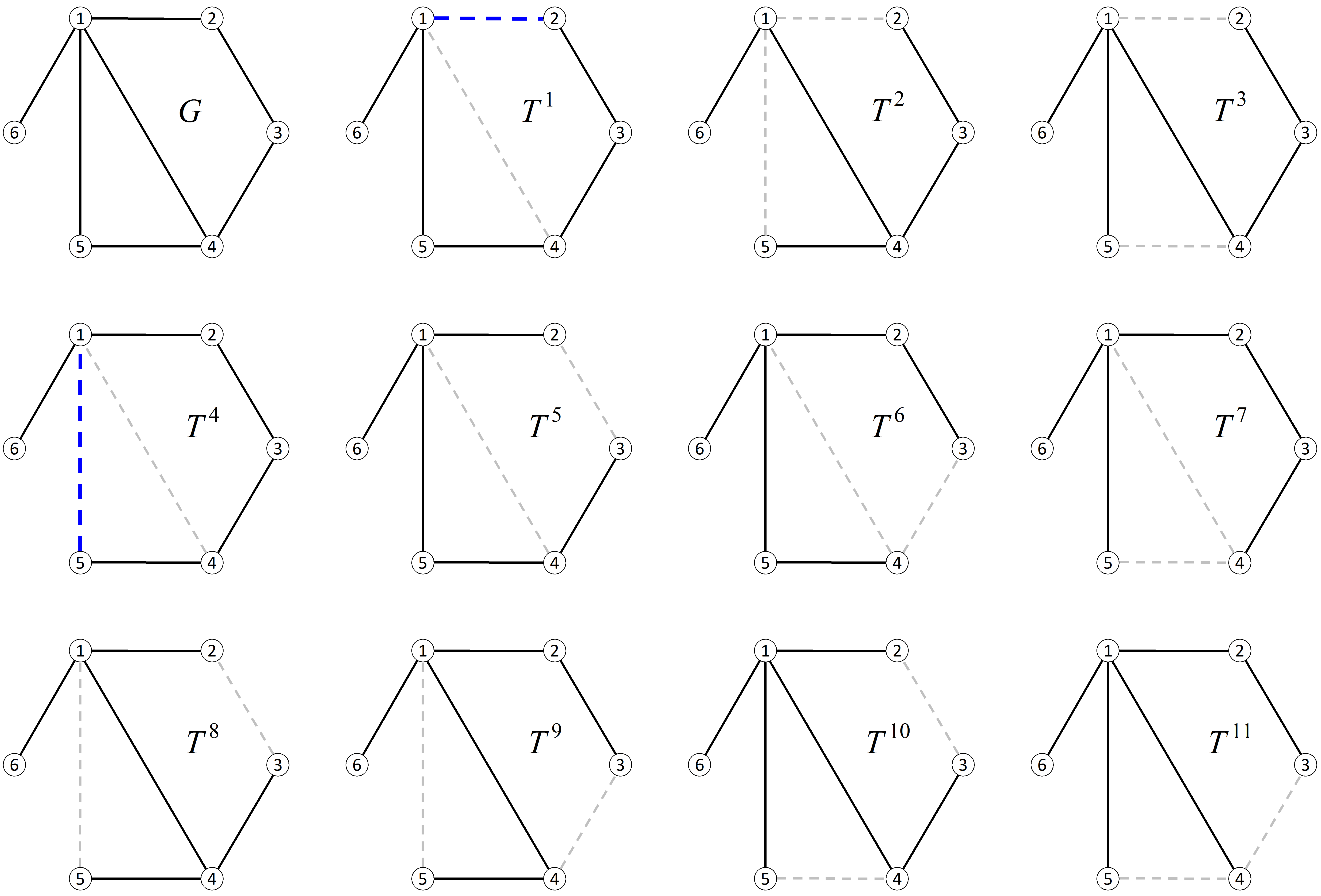

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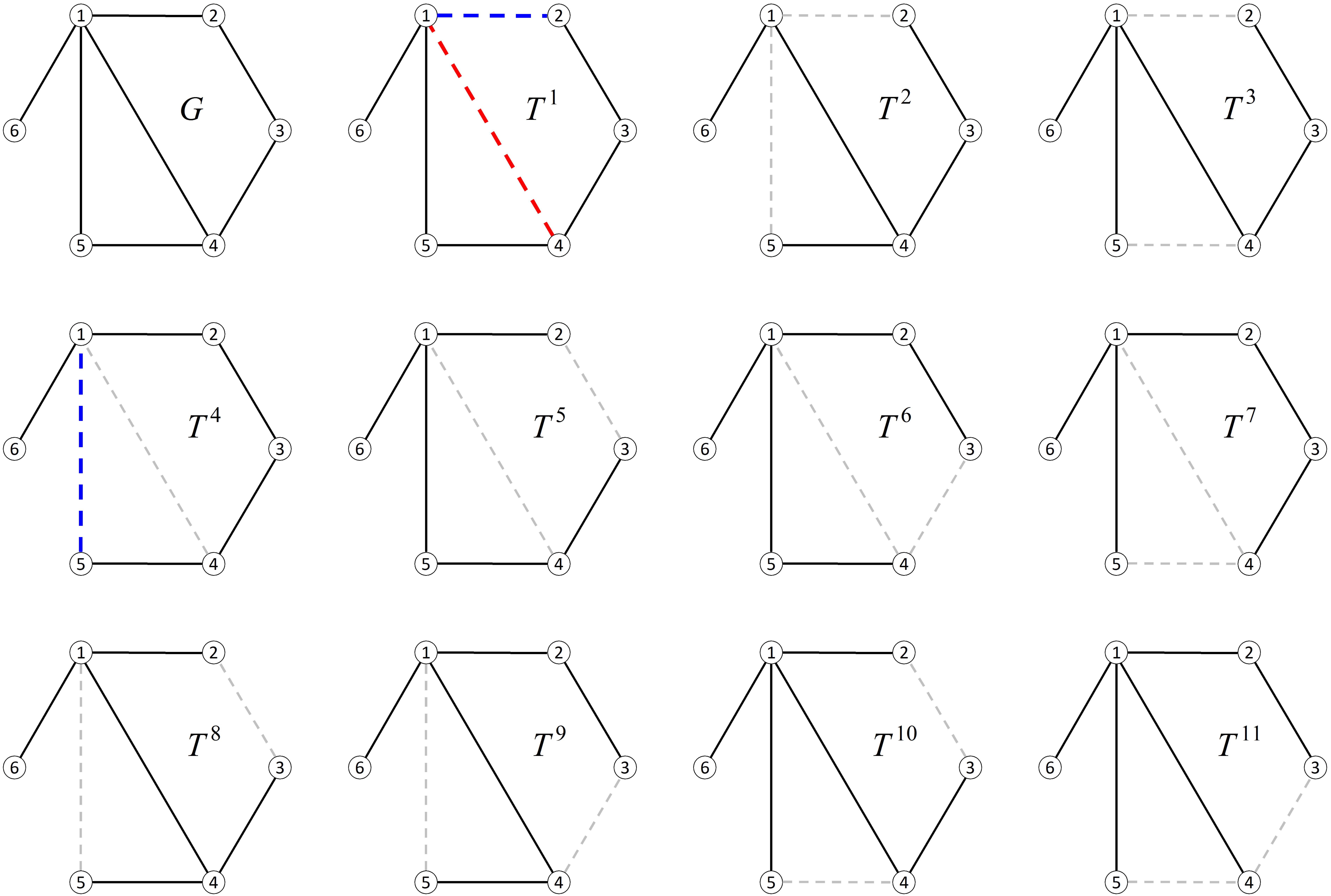

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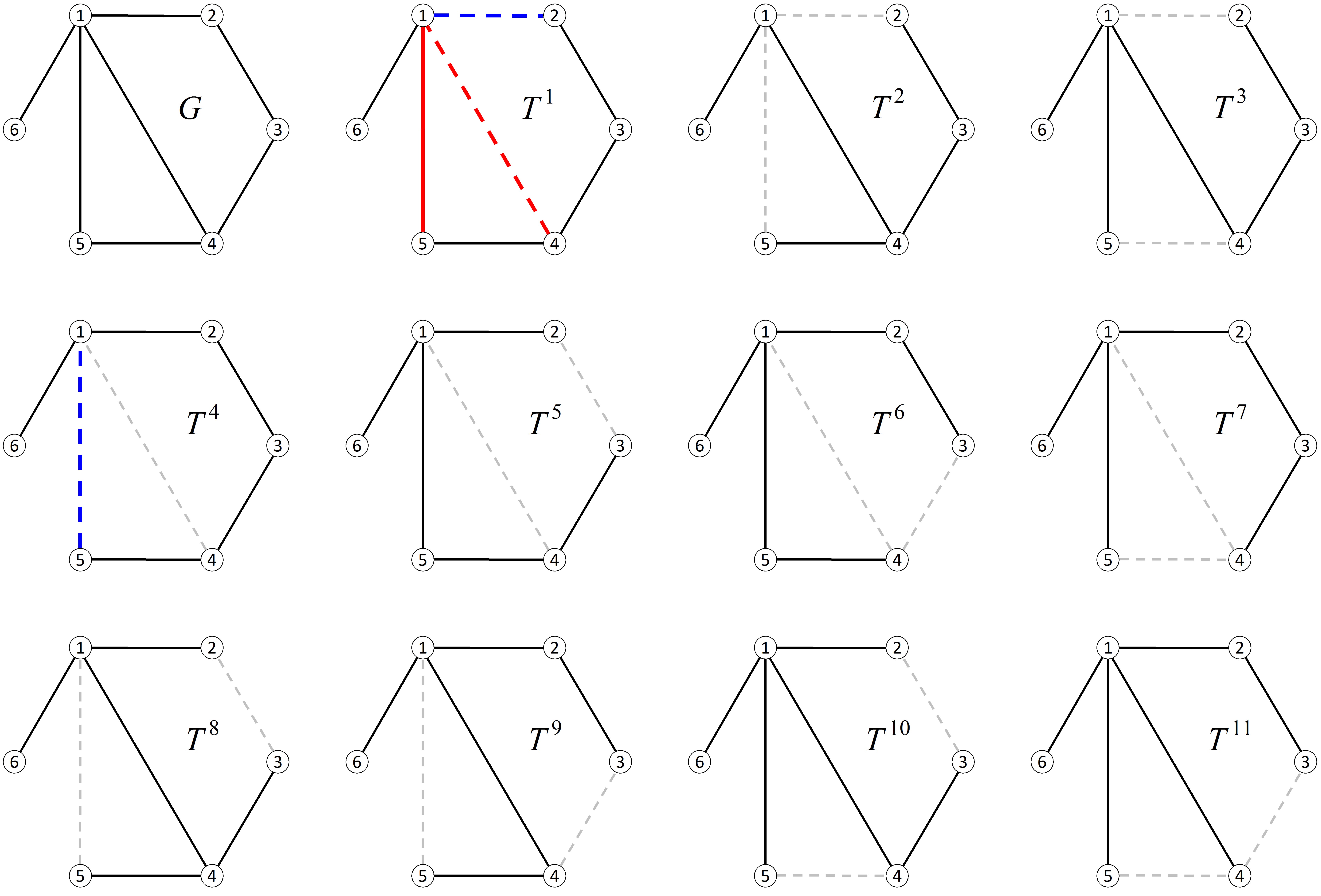

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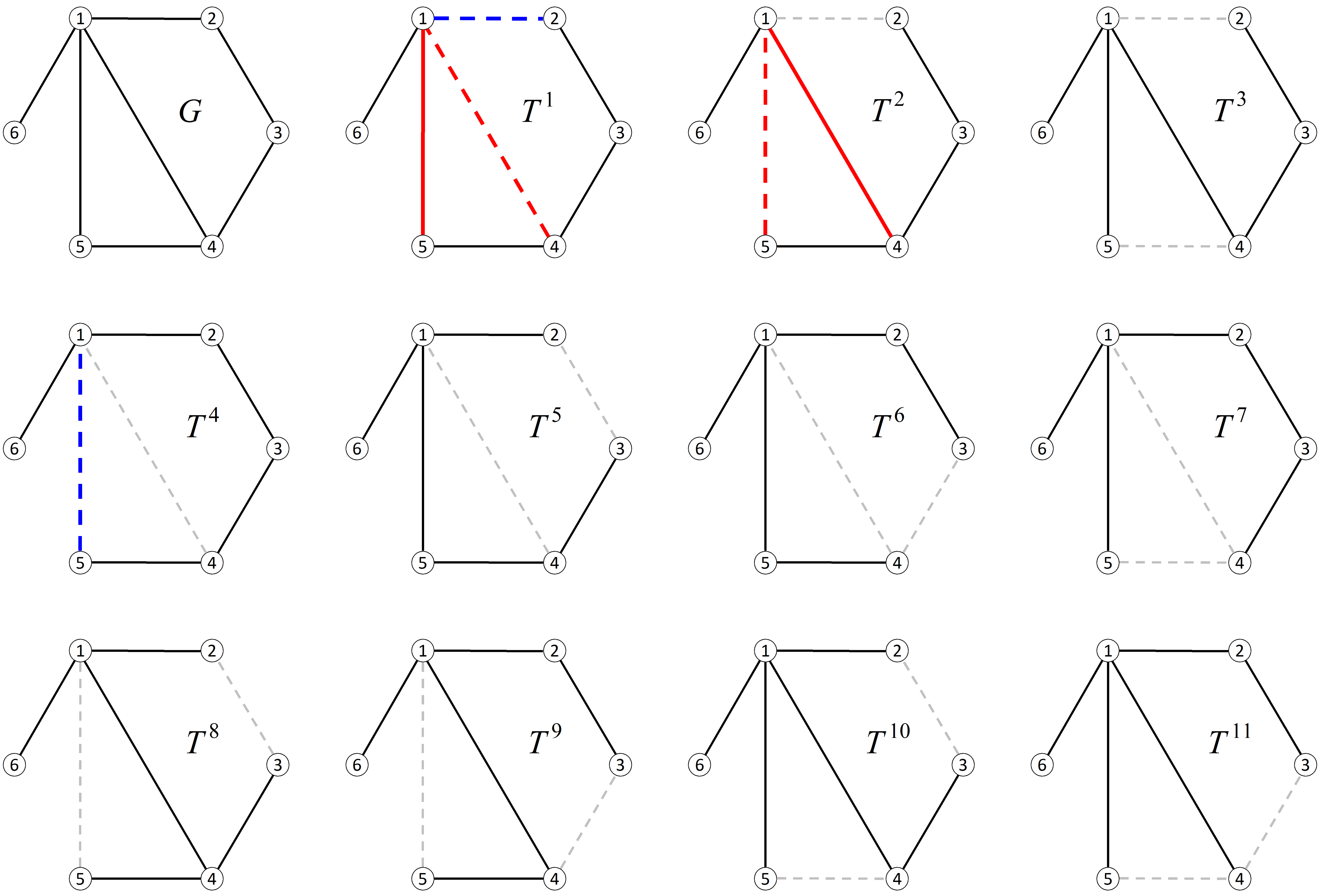
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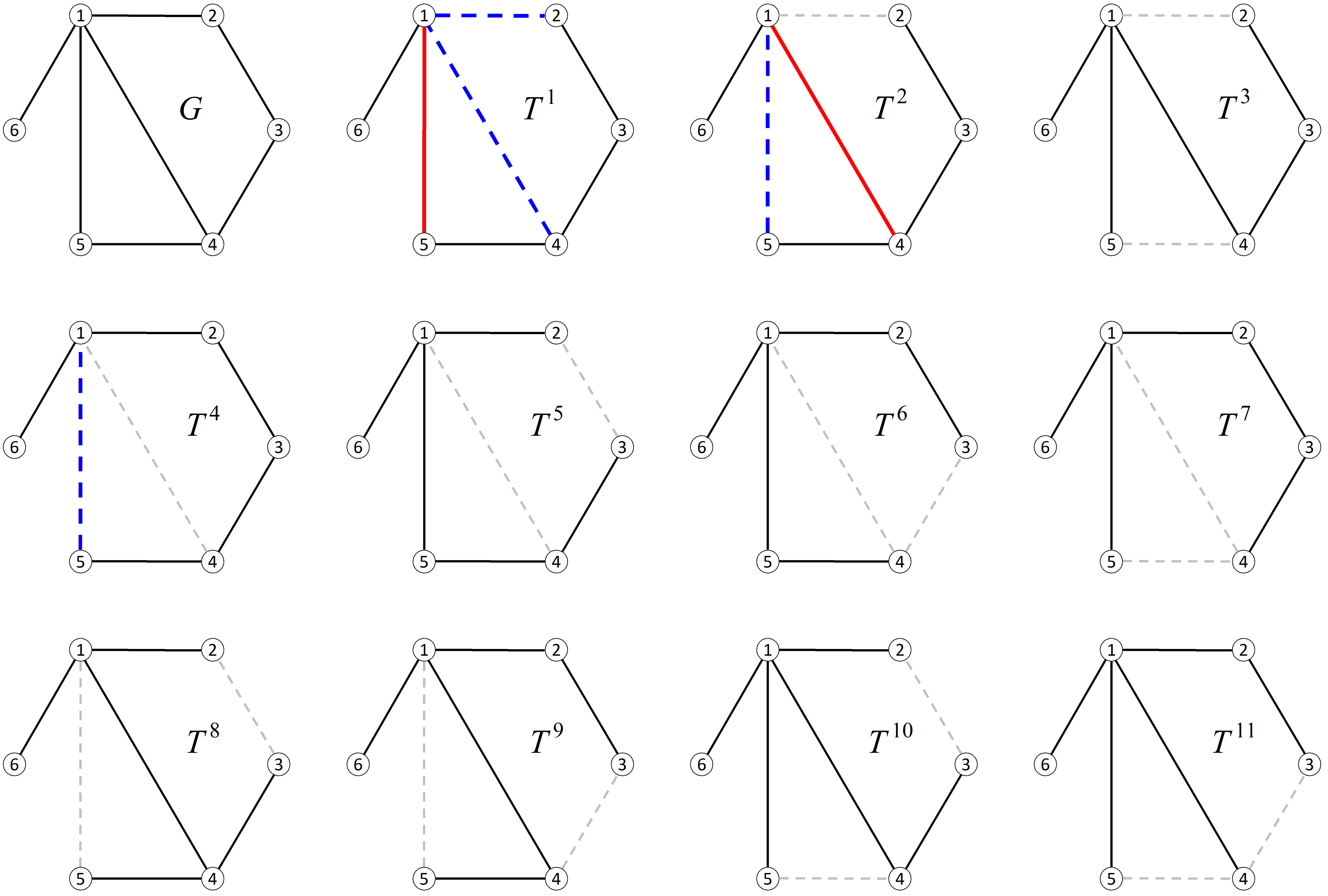

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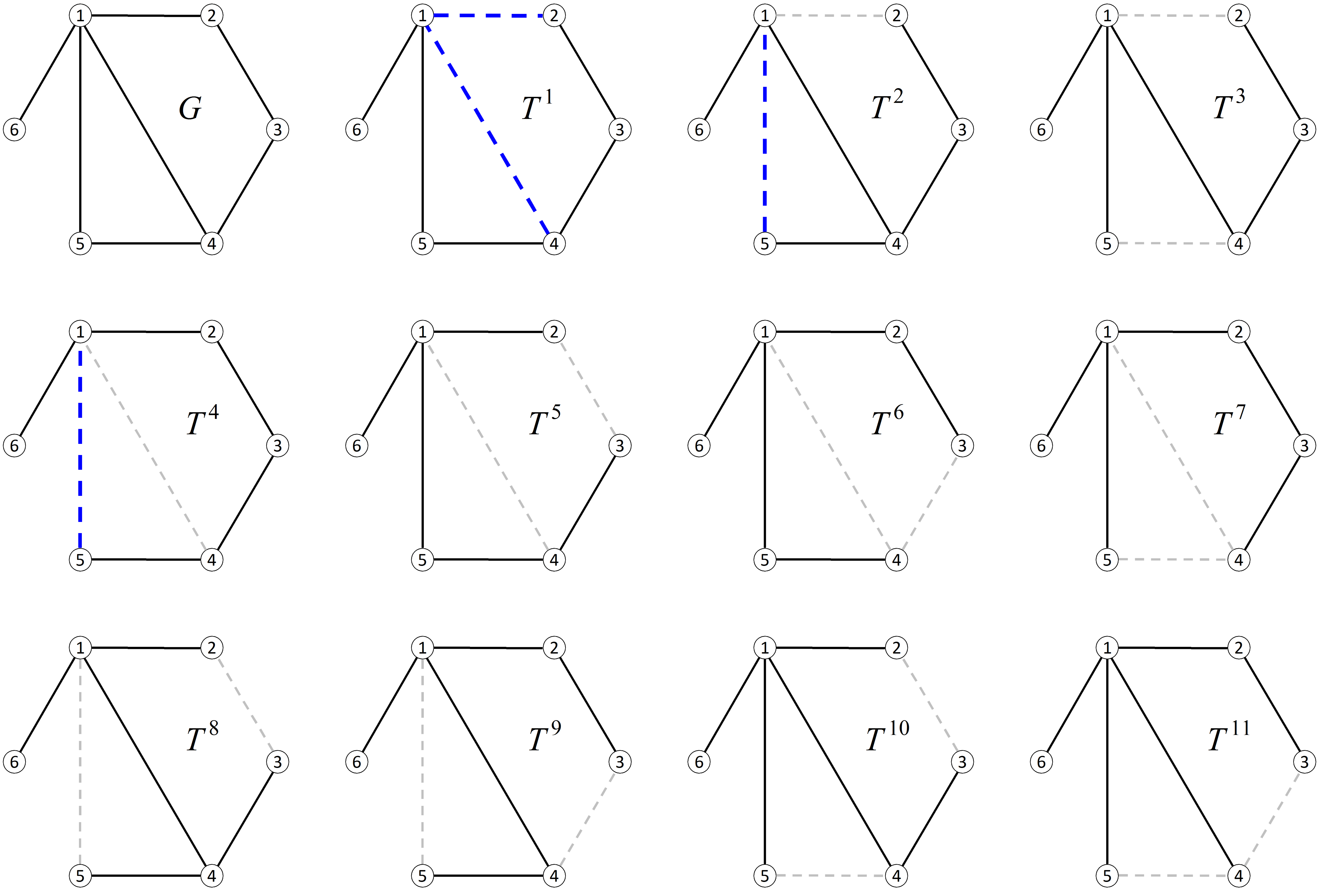
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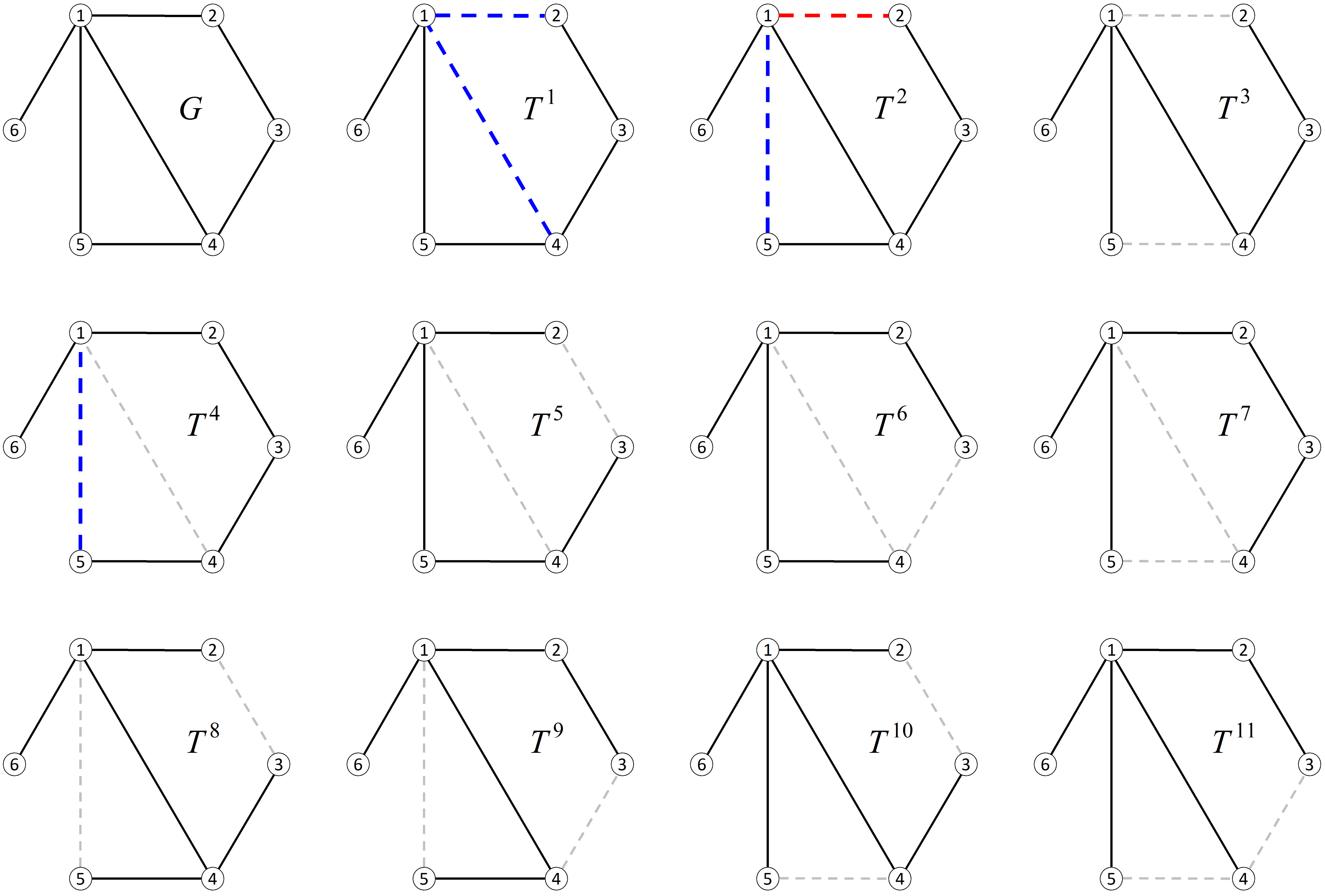

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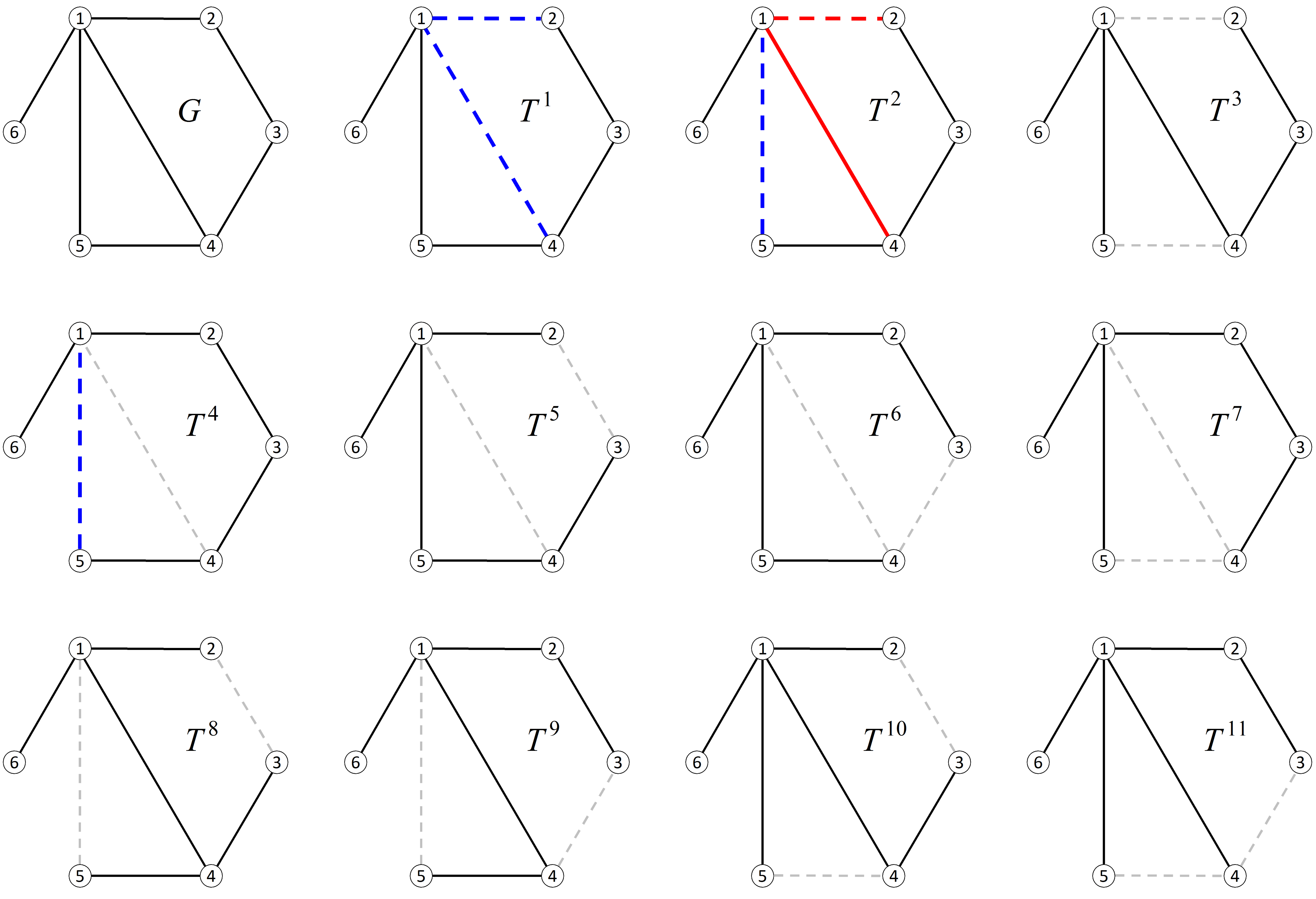
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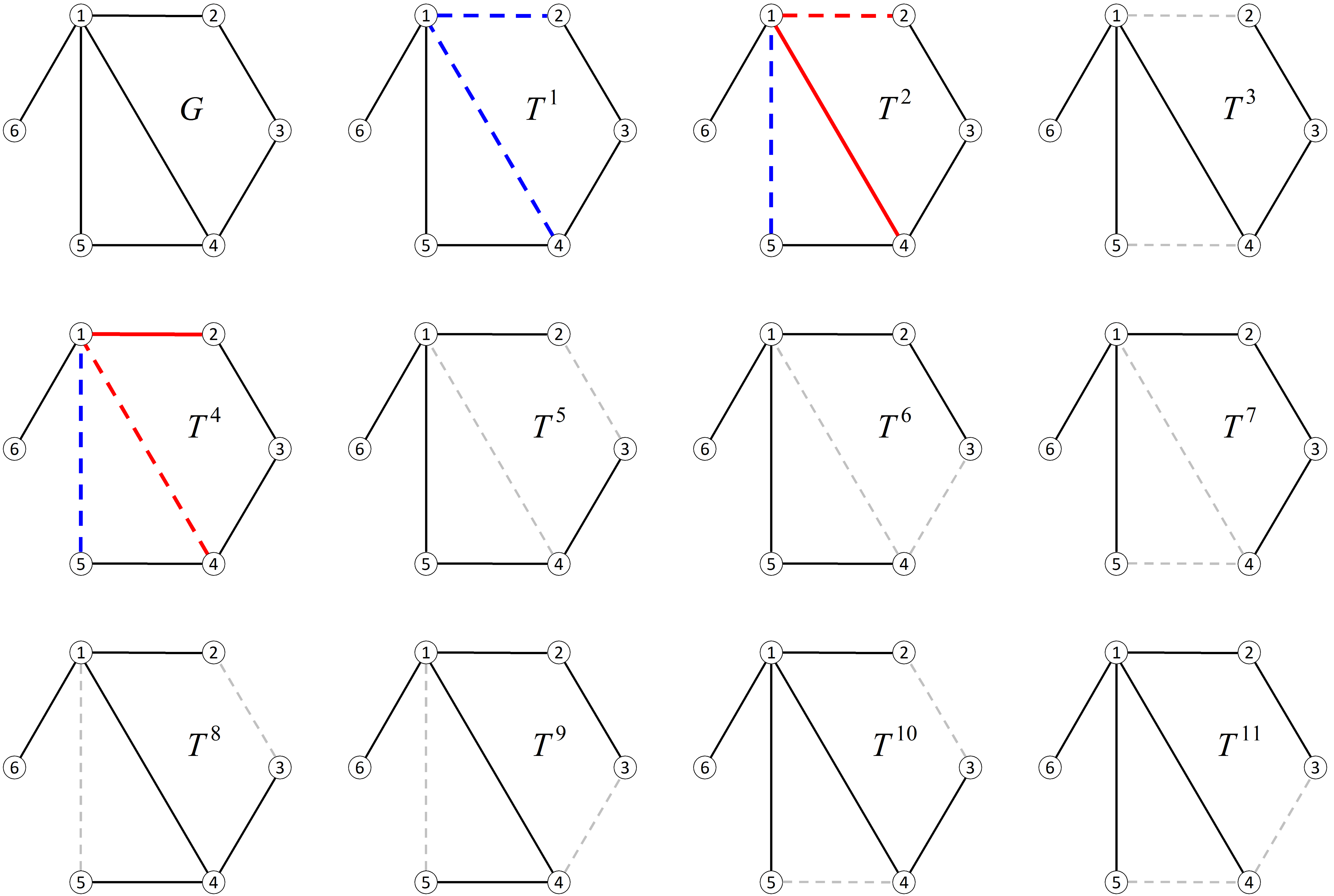

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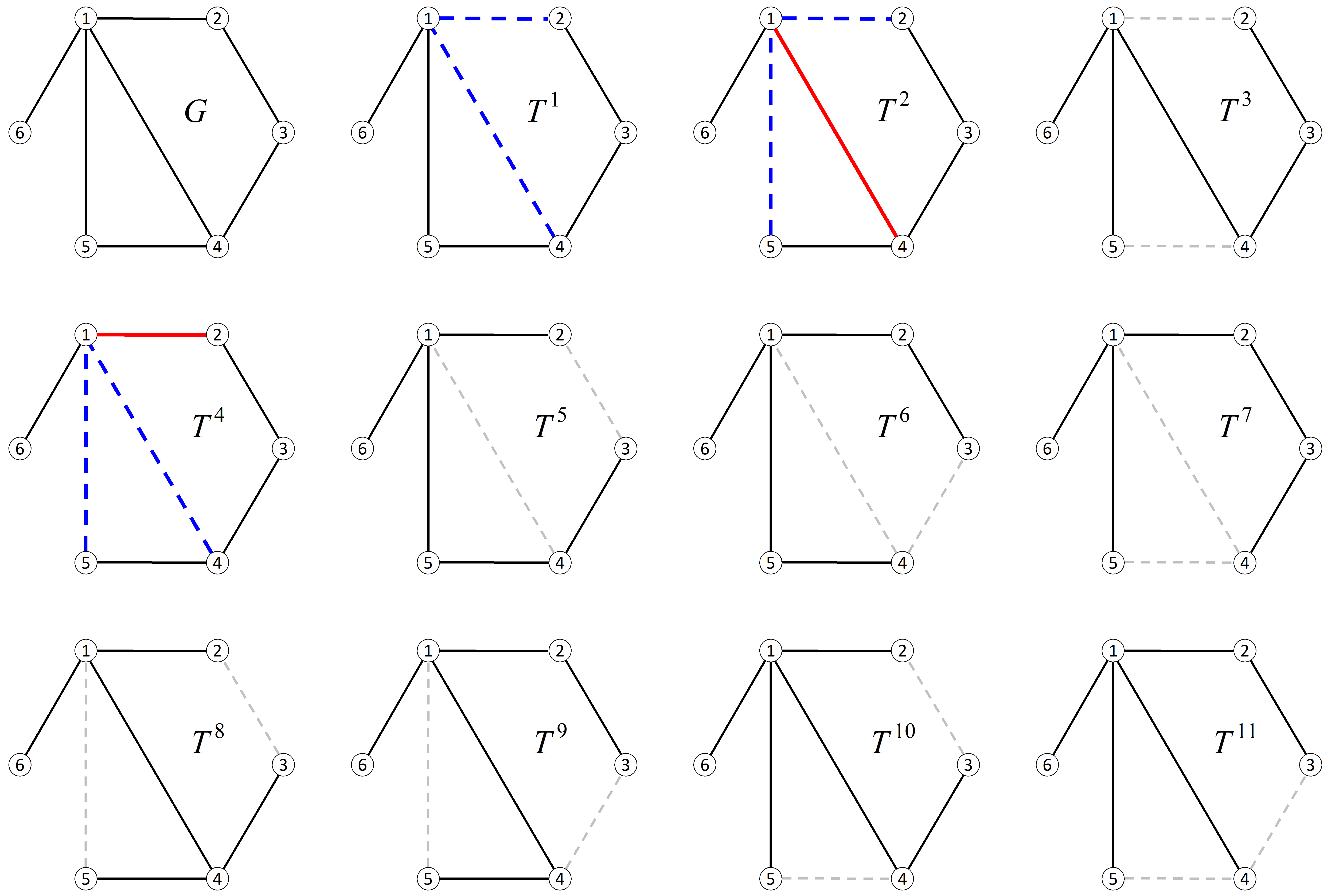

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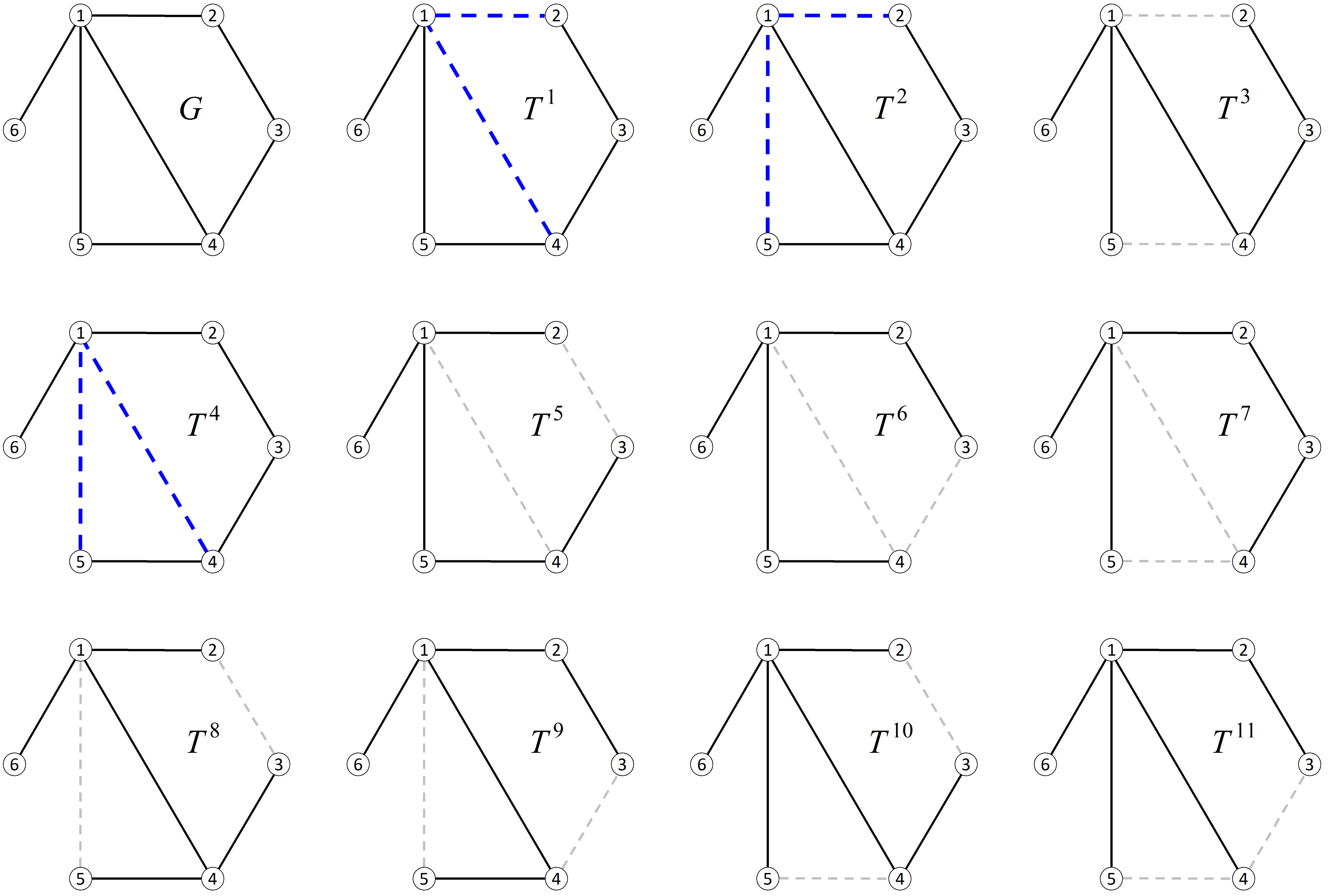

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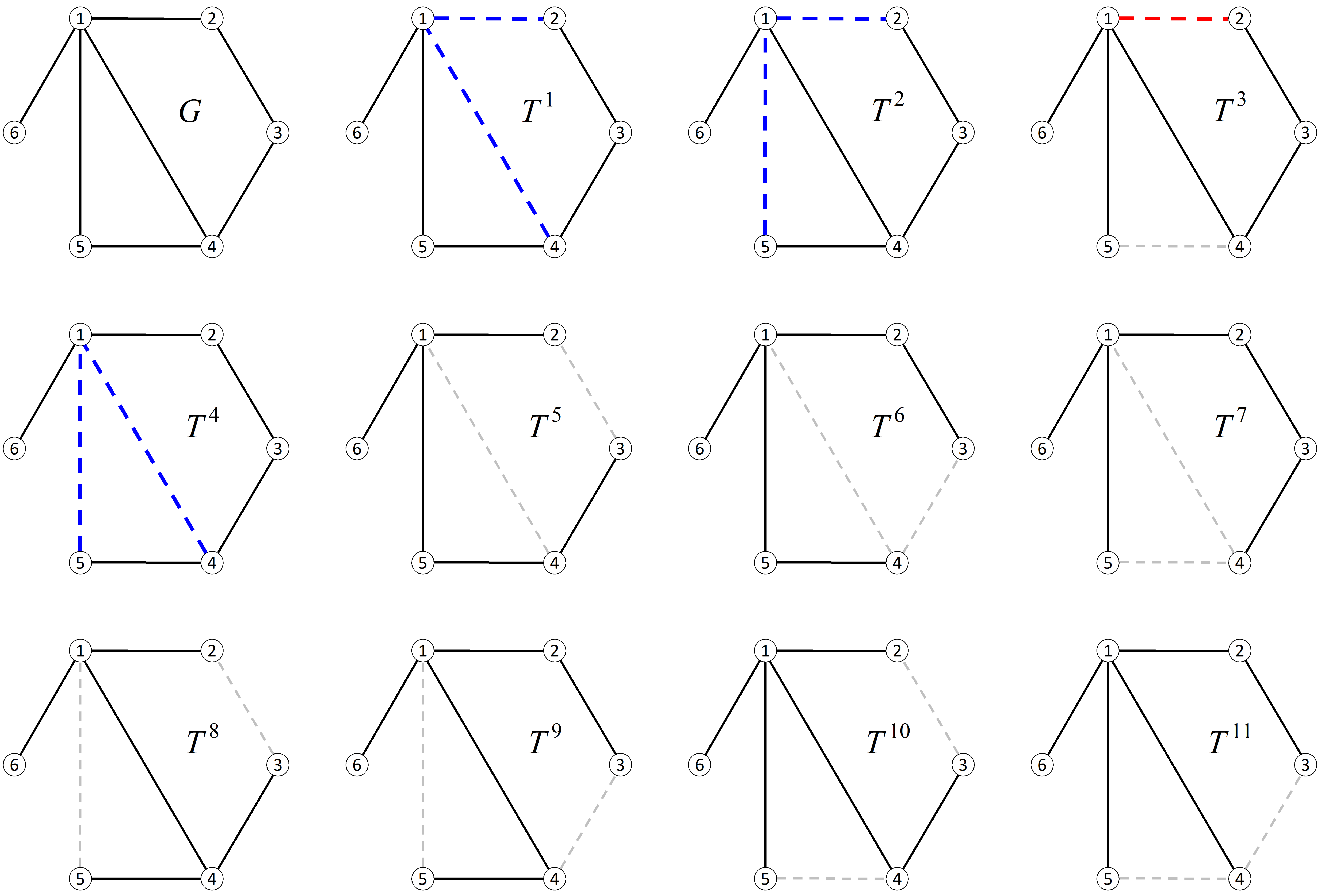
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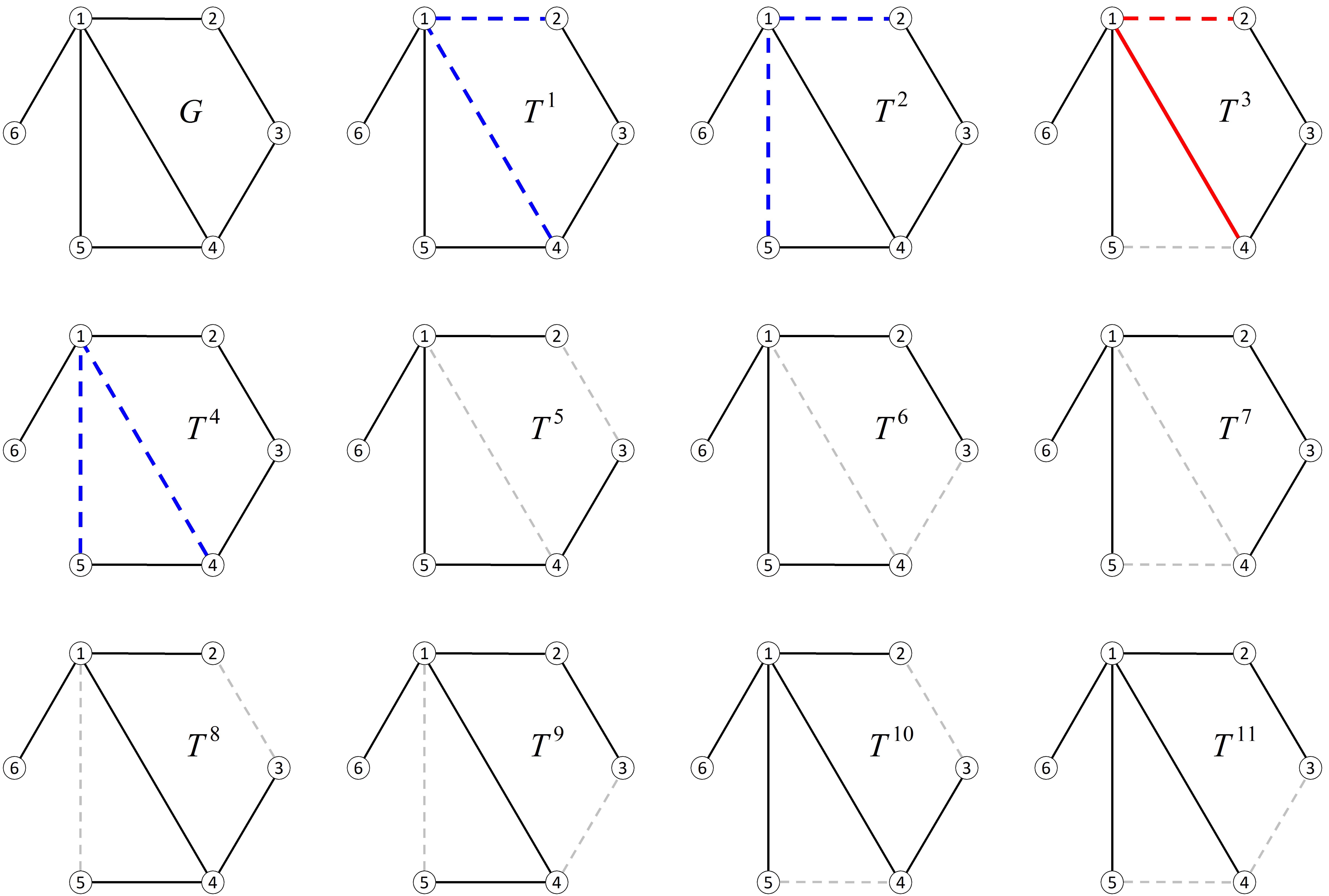

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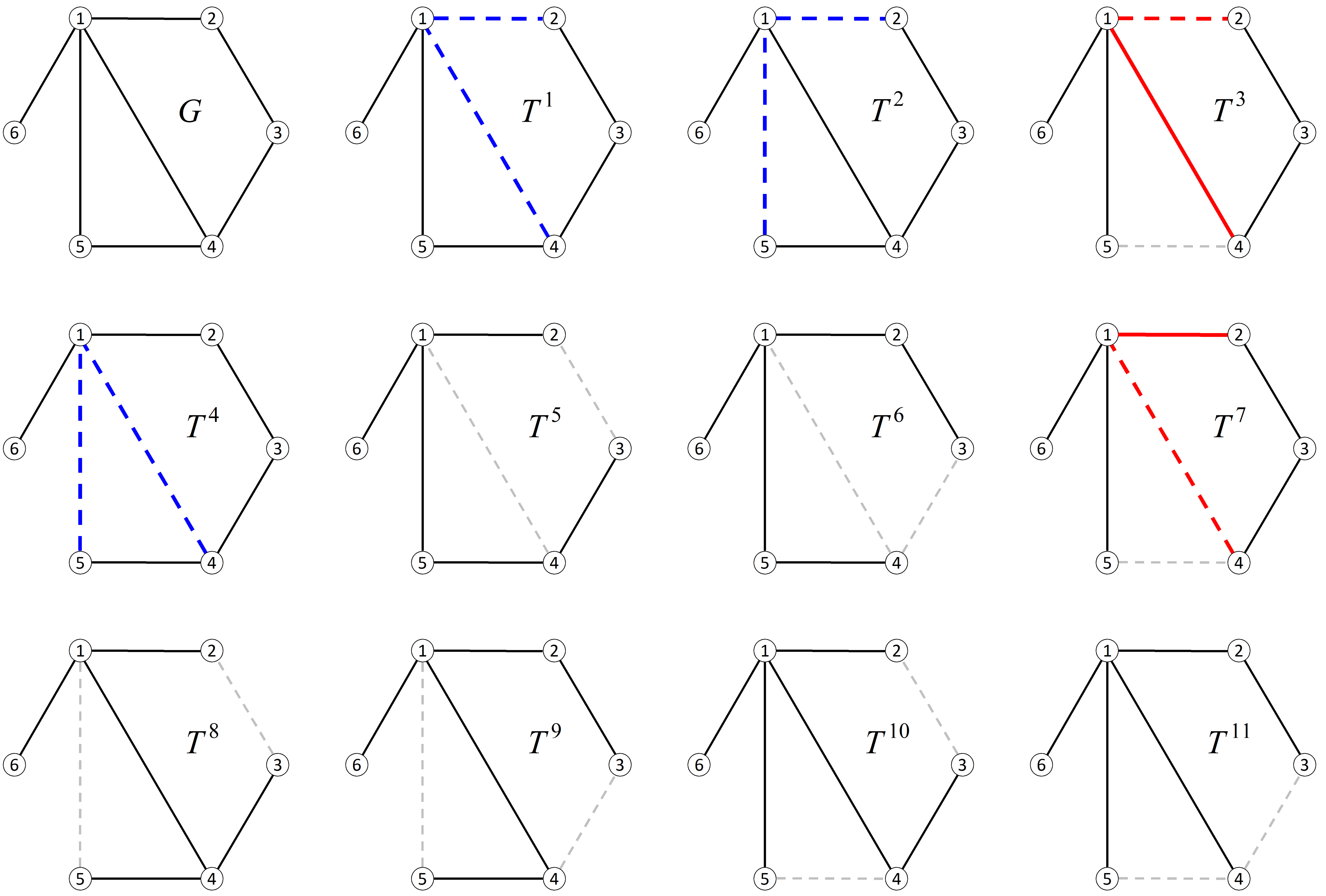

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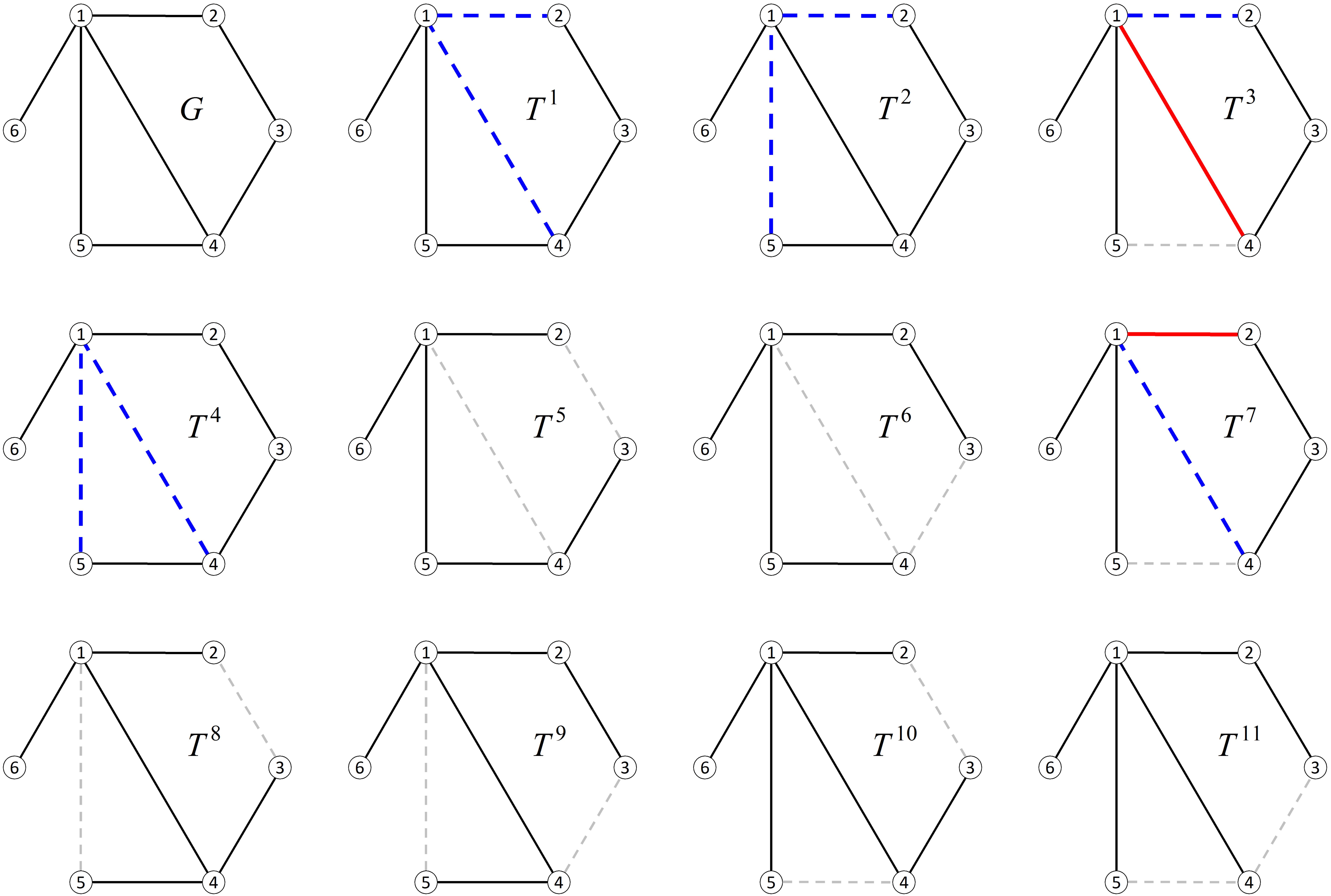

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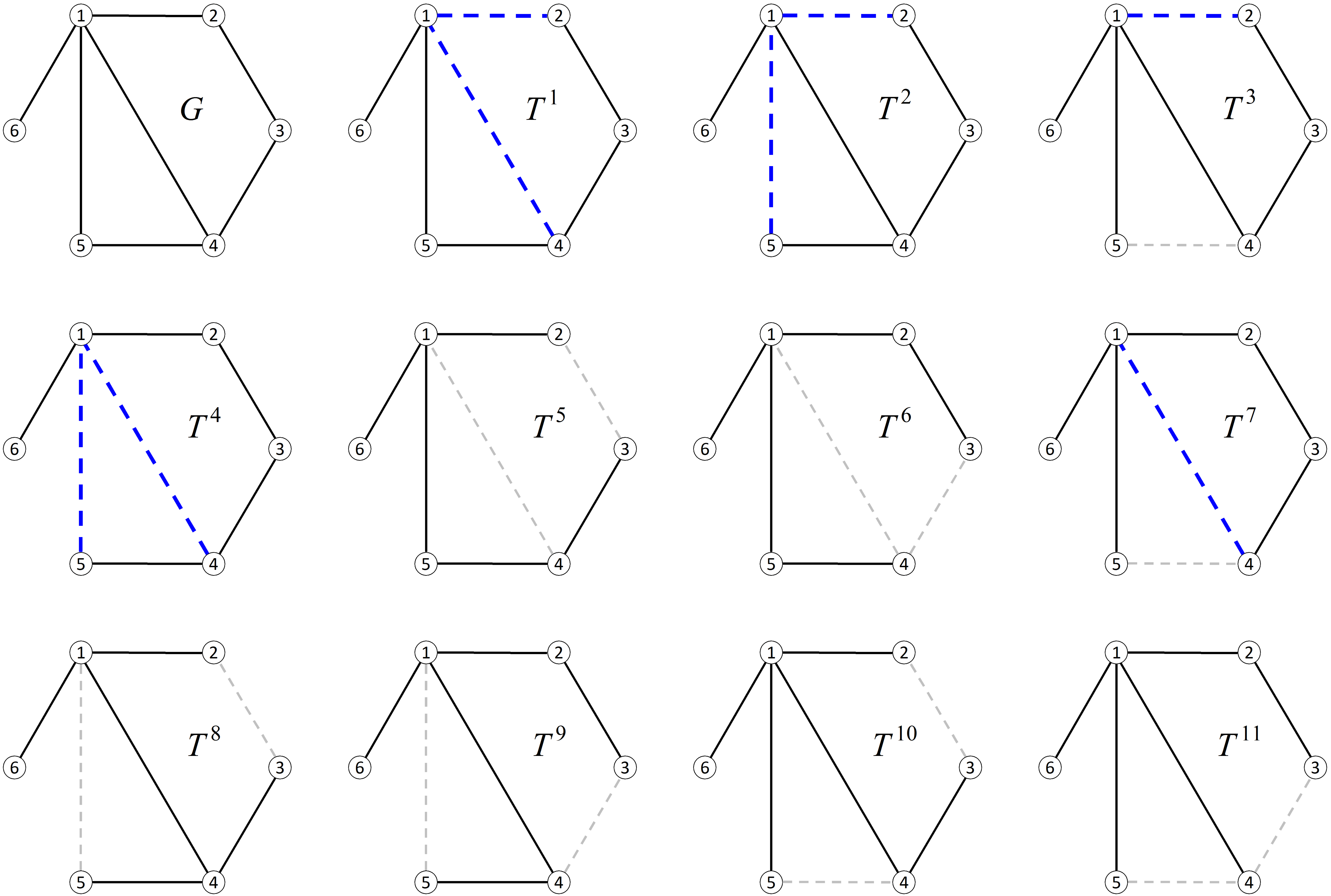

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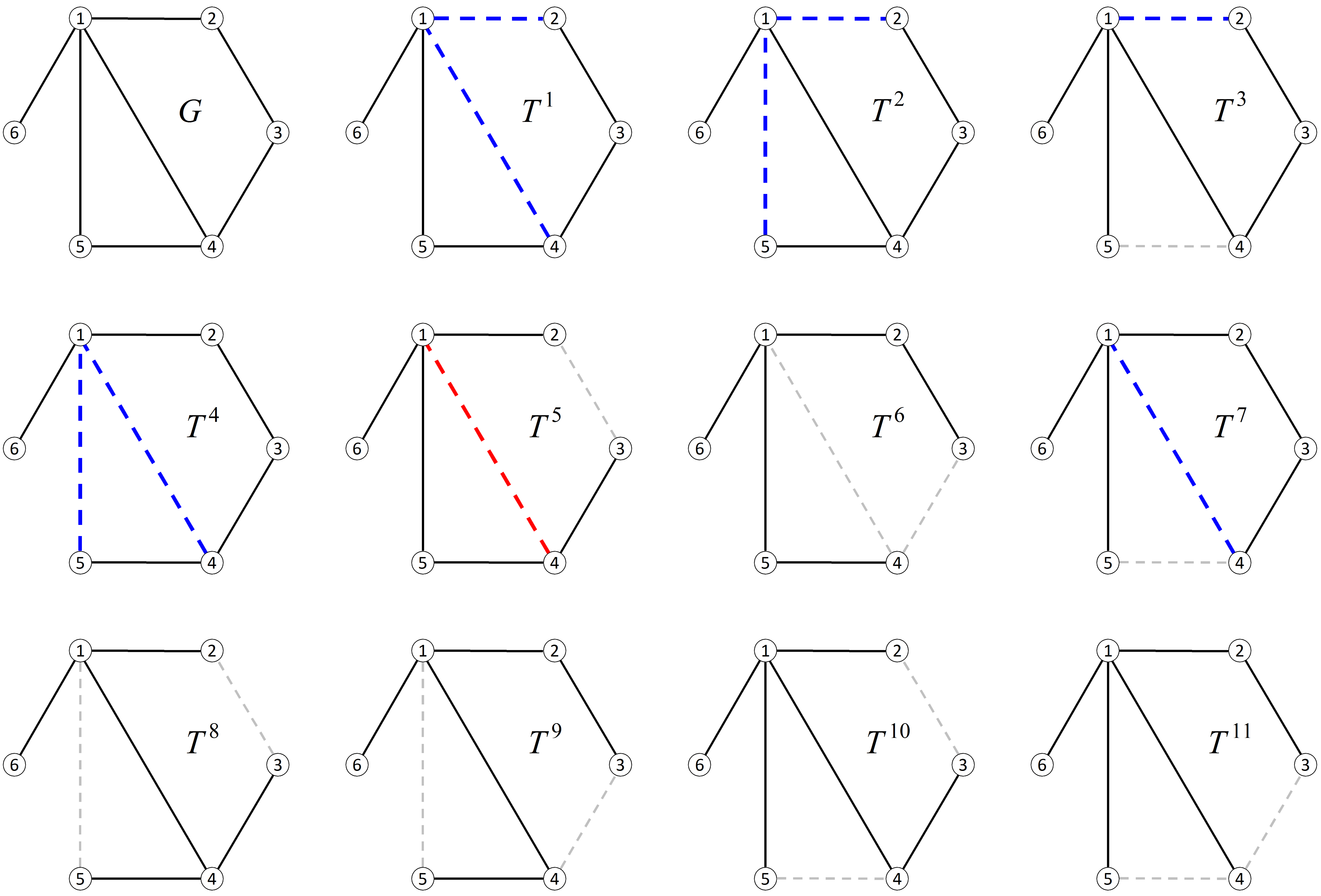

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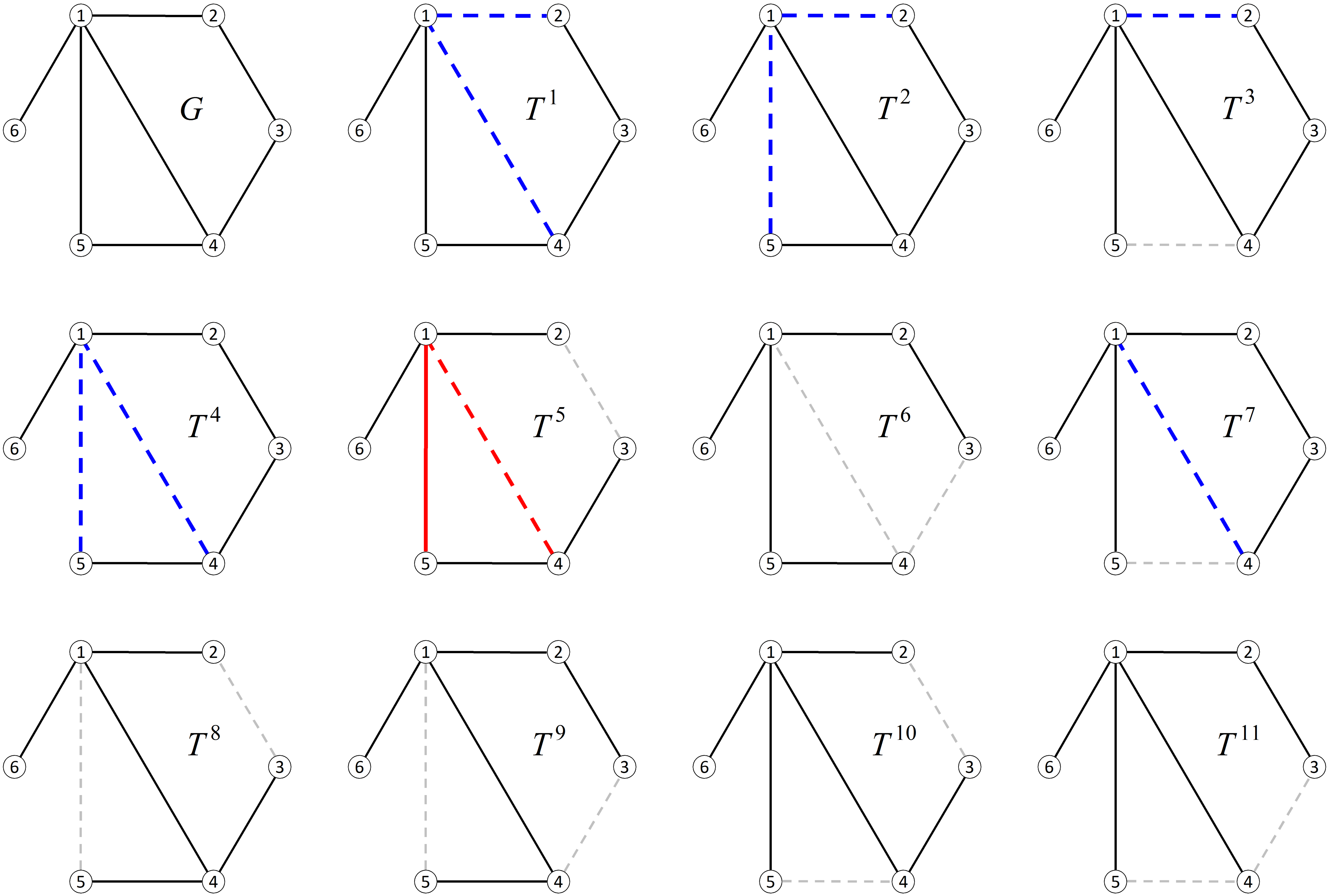

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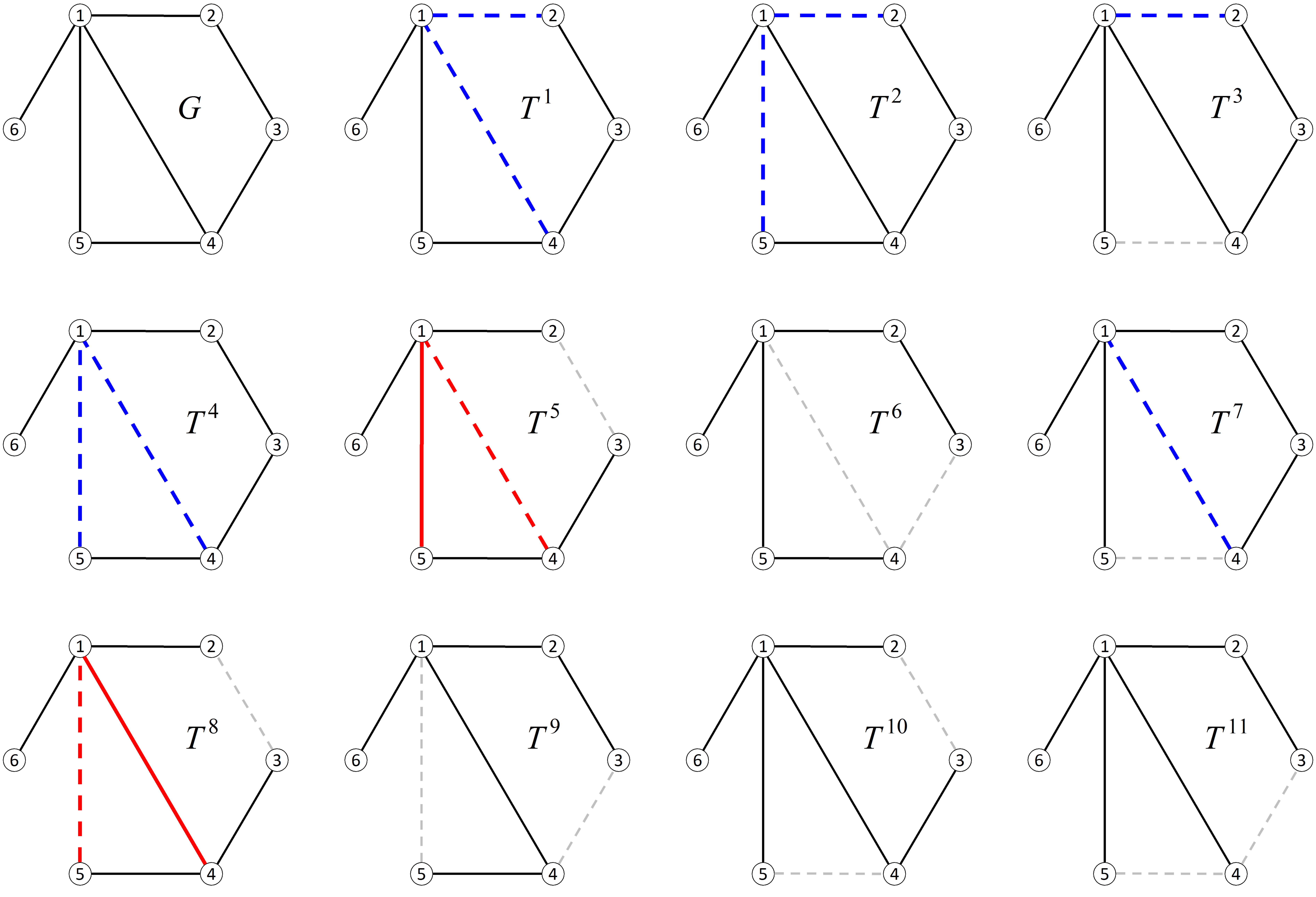

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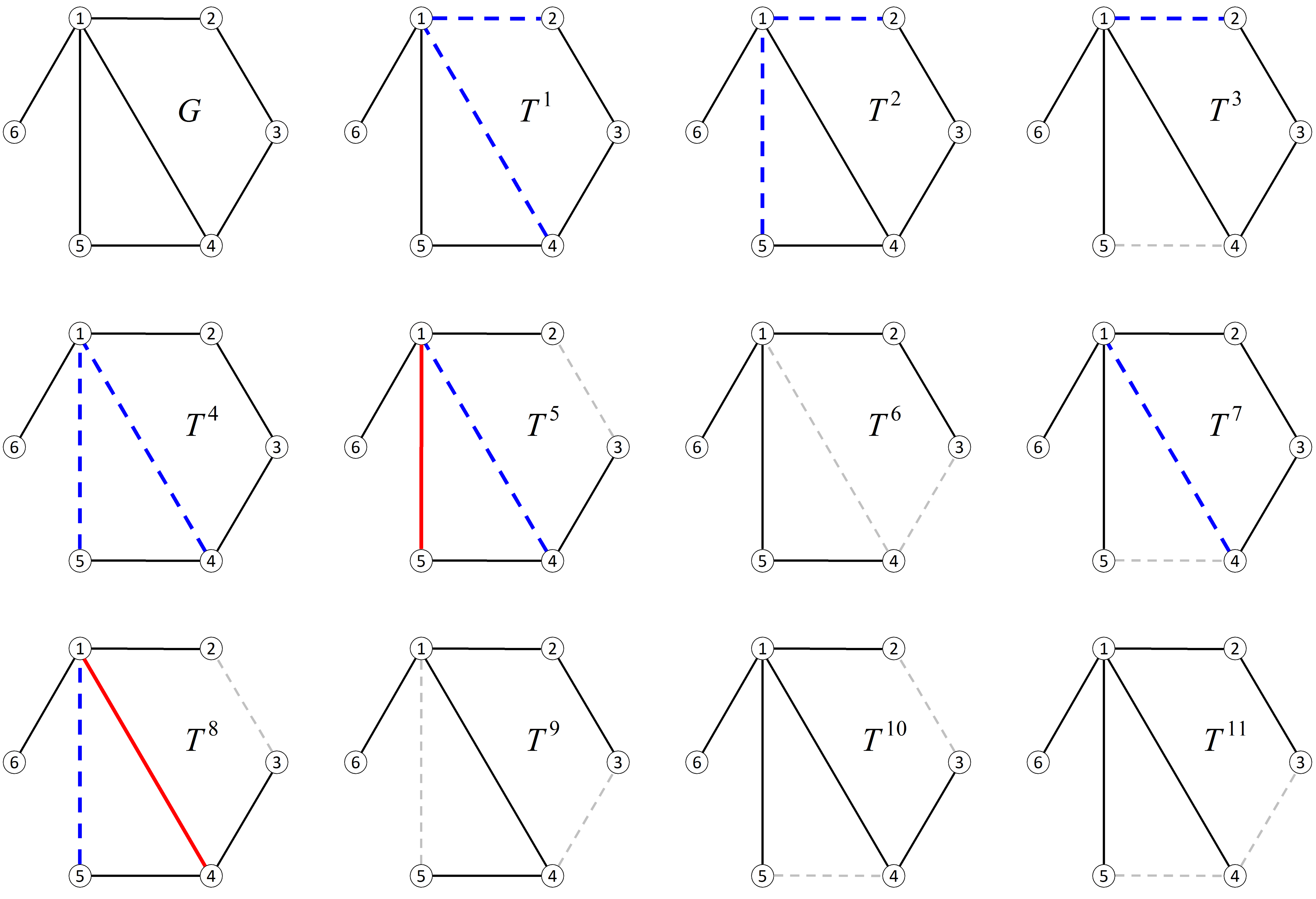

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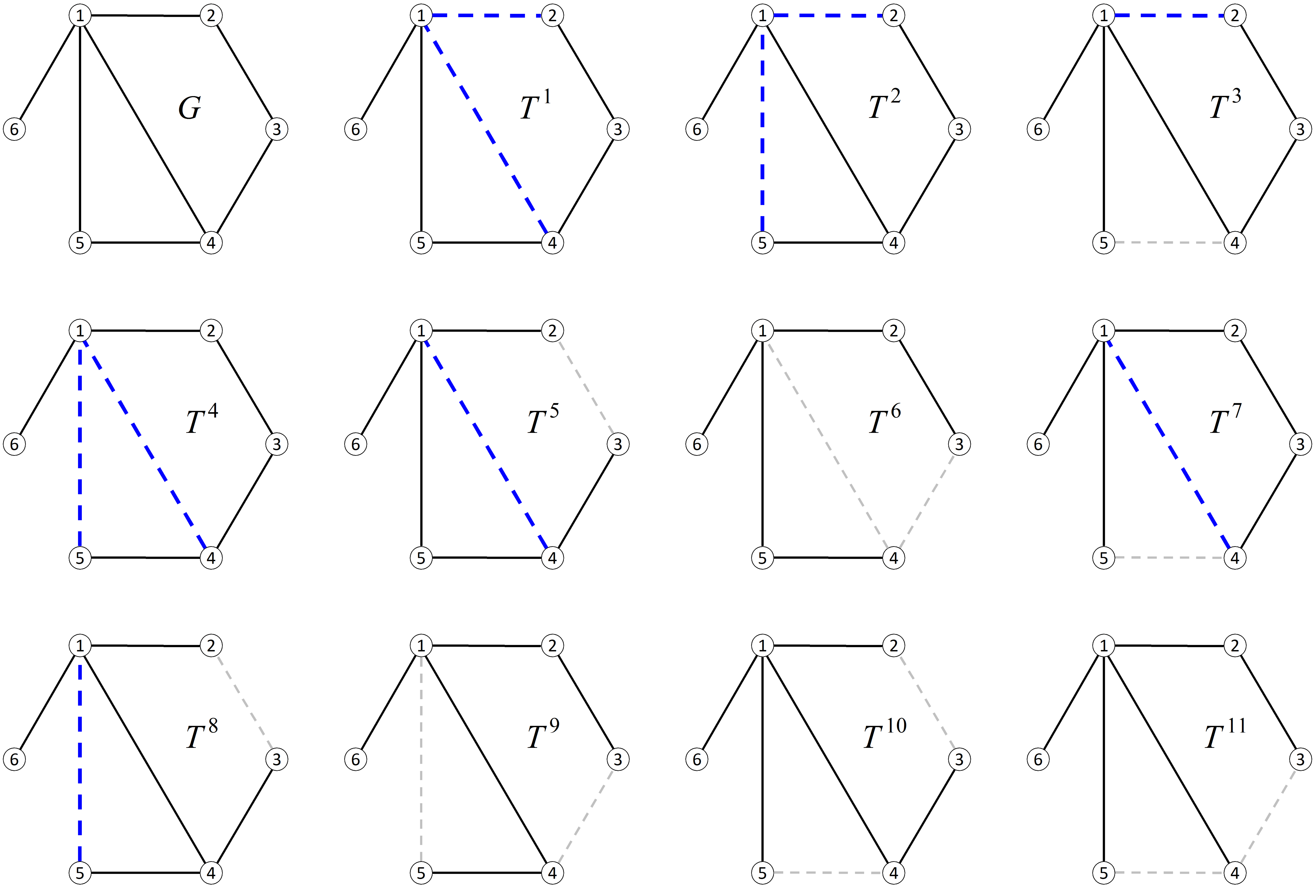
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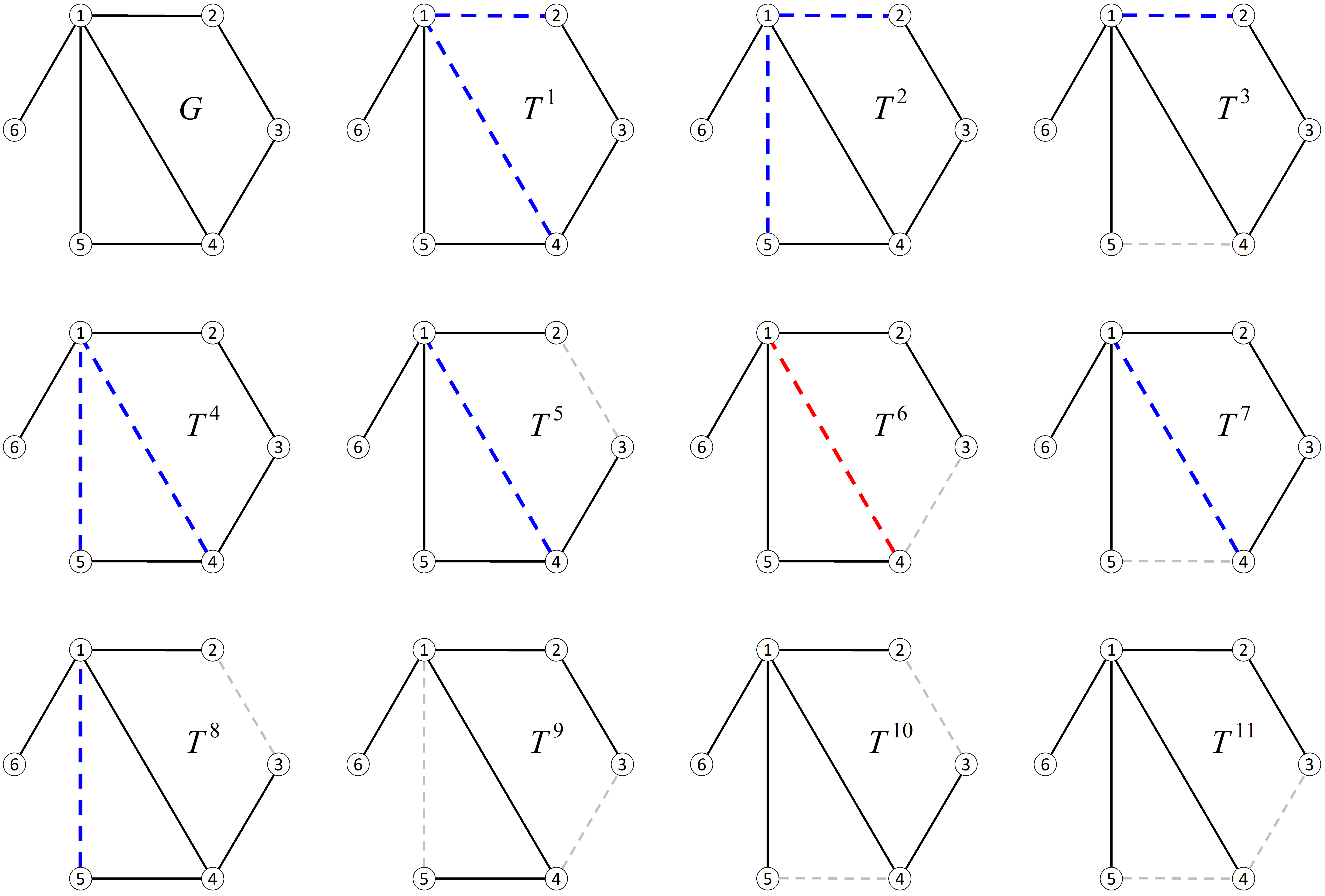
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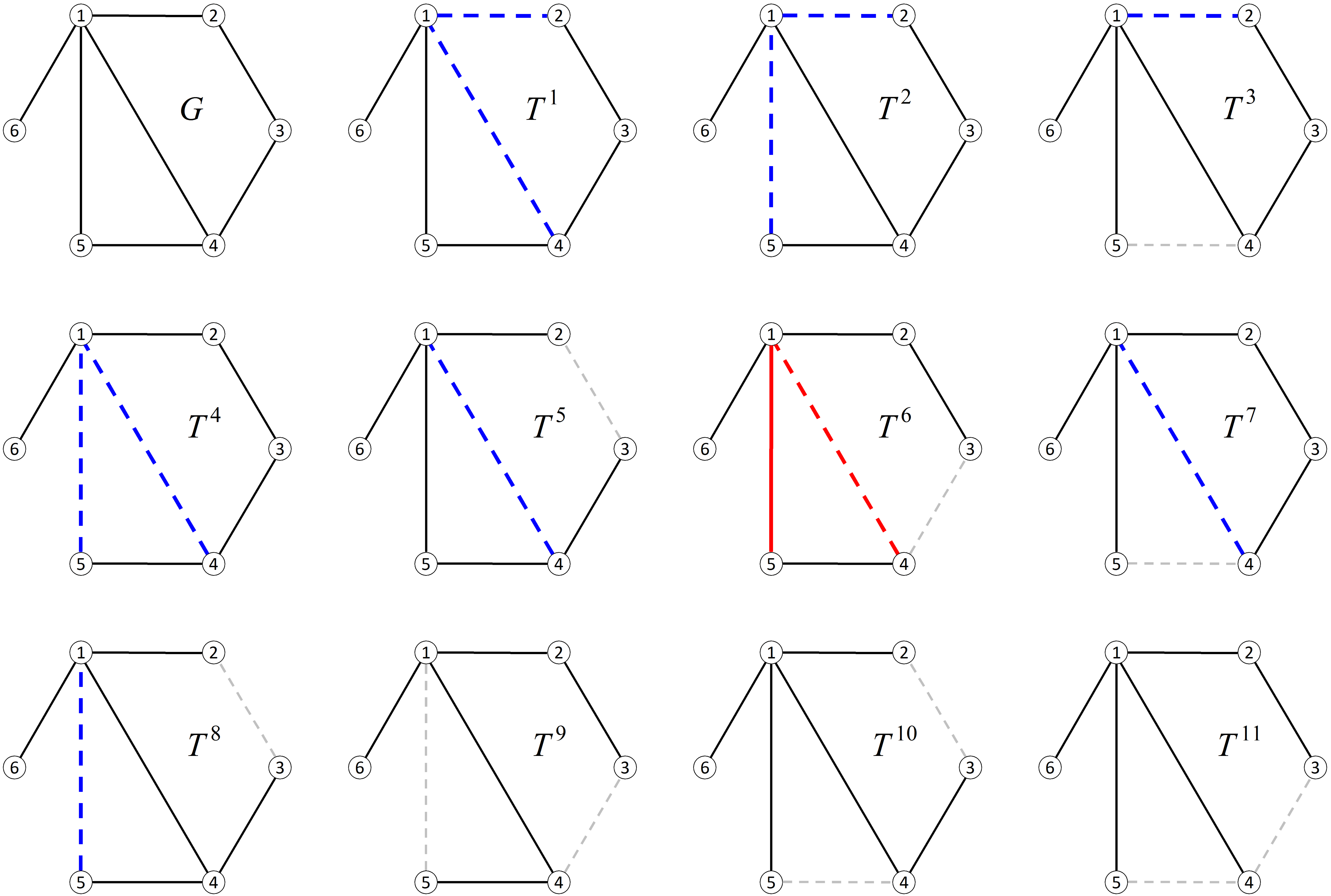

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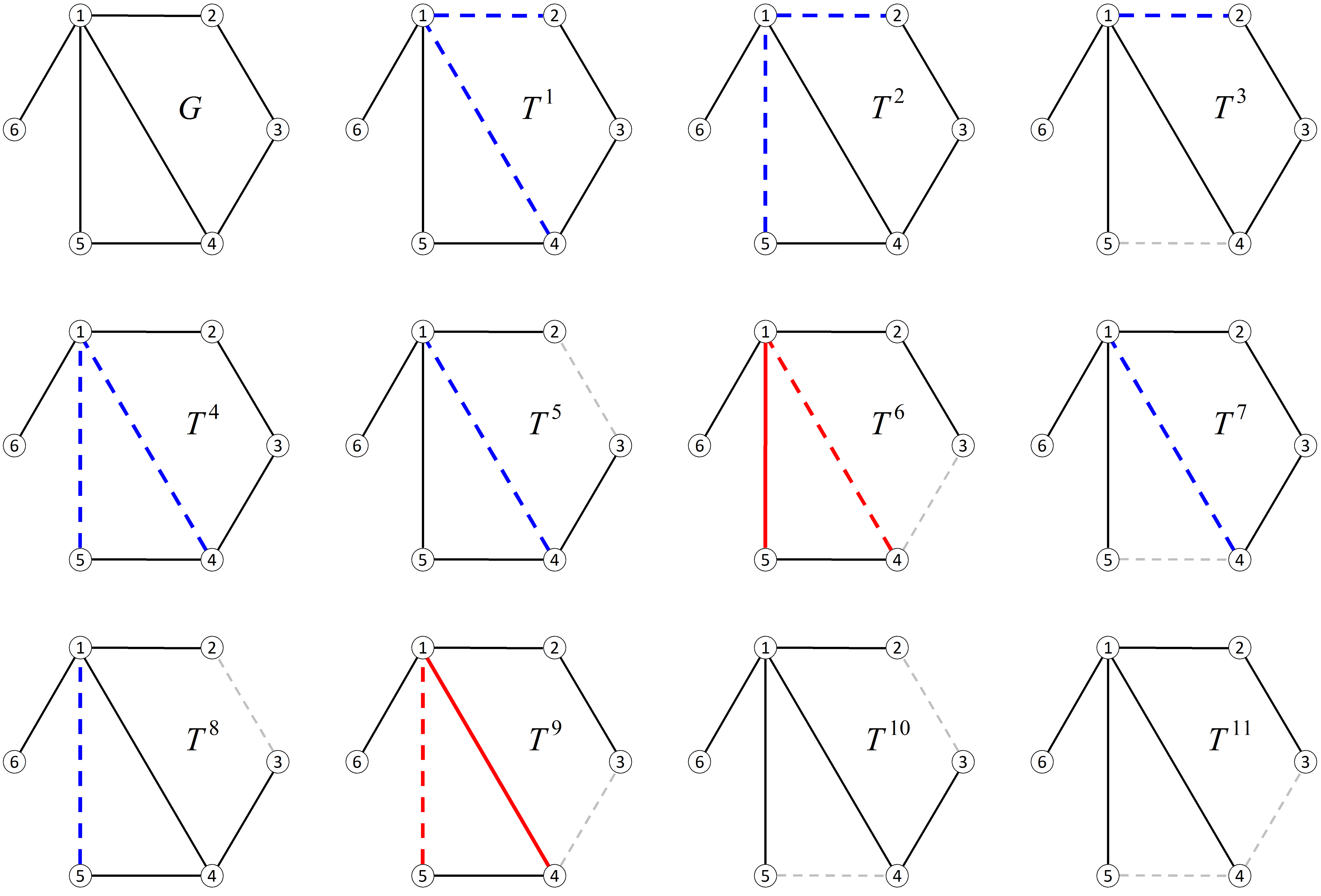
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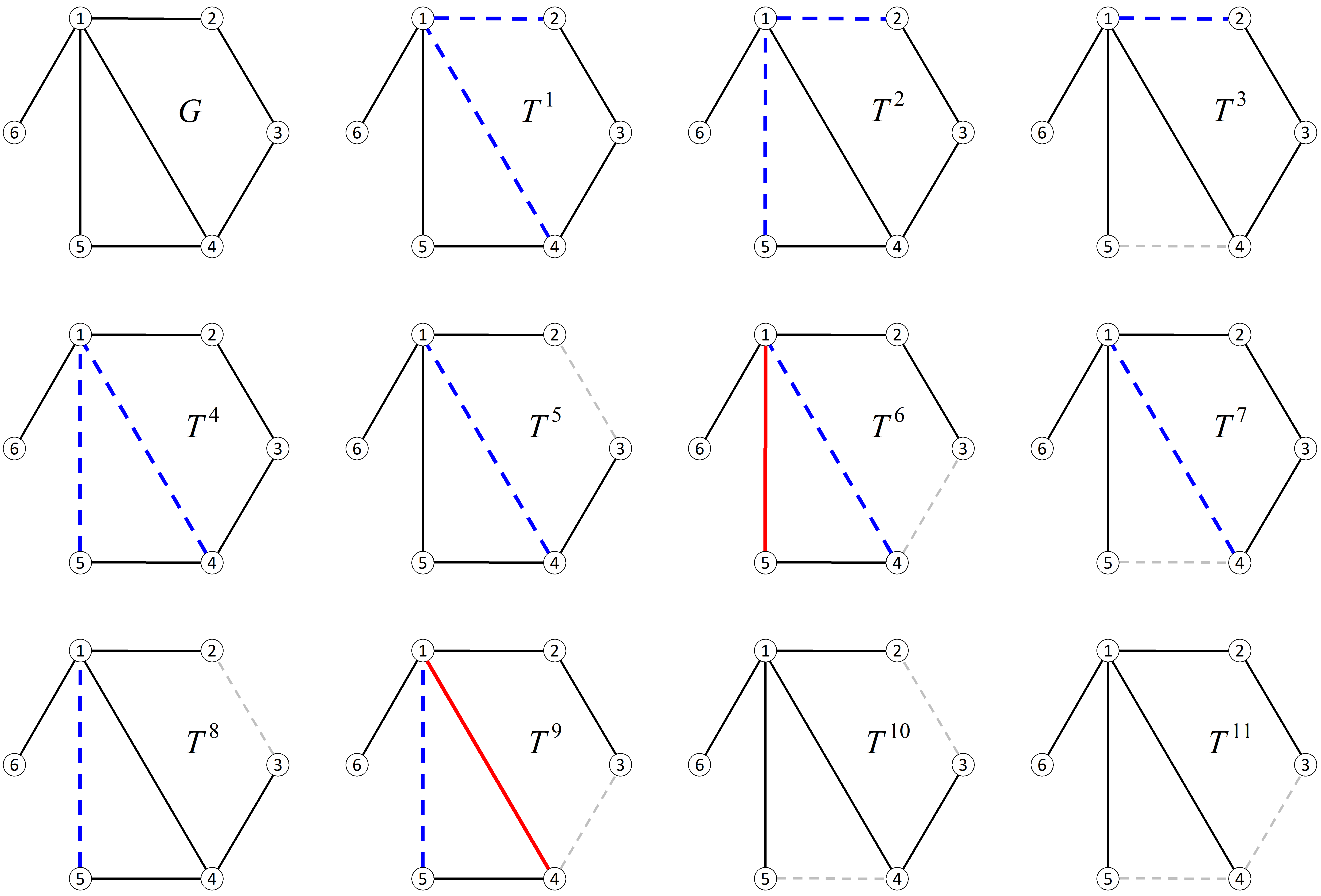
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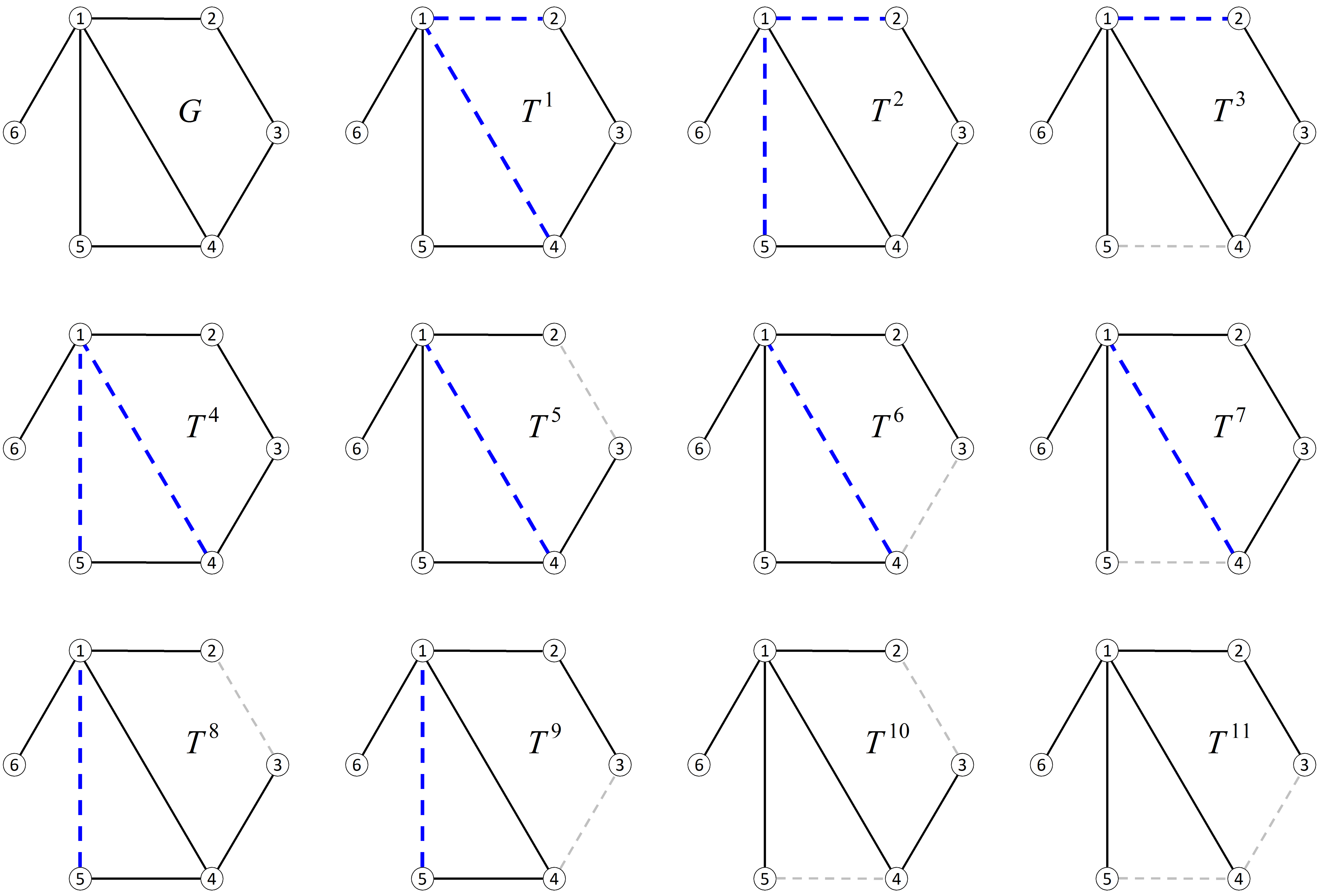
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proof

Finally, to complete the proof, take the sum of equations

$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^s \quad \text{for all } i = 1, \ldots, n$$

for all $s = 1, 2, \ldots, S$ and apply the lemma

$$\sum_{s=1}^{S} \left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^s \right) = S \sum_{k:e(i,k)\in E(G)} b_{ik}$$

to conclude that $\mathbf{y}^{LLS} = \frac{1}{S}\sum_{s=1}^{S} \mathbf{y}^s$. $\square$

Remarks

Complete pairwise comparison matrices ($S = n^{n-2}$) are included in our theorem as a special case, and our proof can also be considered as a second, and shorter proof of the theorem of Lundy, Siraj and Greco (2017).

Special incomplete cases, investigated by Harker (1987); van Uden (2002); Chen, Kou, Tarn, Song (2015); Bozóki (2017) are also included.

Conclusions

The equivalence of two fundamental weighting methods has been shown.

The advantages of two approaches have been united.

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Thank you for attention.

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