Inefficient weights from pairwise comparison matrices with arbitrarily small inconsistency

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Definitions and notations

$\mathcal{PCM}_n$ denotes the set of pairwise comparison matrices of size $n \times n$.

$\lambda_{\max}(\mathbf{A})$ denotes the dominant eigenvalue of pairwise comparison matrix $\mathbf{A}$ of size $n \times n$.

$\mathbf{w}^{EM(\mathbf{A})}$, also called EM *weight vector*, denotes the right eigenvector of $\mathbf{A}$ corresponding to $\lambda_{\max}(\mathbf{A})$.

$\mathbf{w}^{EM(\mathbf{A})}$ is usually normalized to 1, that is, $\sum_{i=1}^{n} w_i^{EM(\mathbf{A})} = 1.$

$\mathbf{X}^{EM(\mathbf{A})} = \mathbf{X}^{EM} \stackrel{\text{def}}{=} \left[\frac{w_i^{EM(\mathbf{A})}}{w_j^{EM(\mathbf{A})}} \right]_{i,j=1,\ldots,n}$ is the consistent pairwise comparison matrix generated by $\mathbf{w}^{EM(\mathbf{A})}$.
It is the approximation of $\mathbf{A}$ by the eigenvector method.

Inconsistency index CR

Saaty defined the inconsistency index as

$$CR(\mathbf{A}) \stackrel{\text{def}}{=} \frac{\frac{\lambda_{\max}(\mathbf{A})-n}{n-1}}{\frac{\overline{\lambda_{\max}^{n\times n}}-n}{n-1}} = \frac{\lambda_{\max}(\mathbf{A})-n}{\overline{\lambda_{\max}^{n\times n}}-n},$$

where $\overline{\lambda_{\max}^{n\times n}}$ denotes the average value of the maximal eigenvalue of randomly generated pairwise comparison matrices of size $n \times n$ such that each element a_{ij} $(i < j)$ is chosen from the ratio scale $1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 9$ with equal probability. $CR(\mathbf{A})$ is a positive linear transformation of $\lambda_{\max}(\mathbf{A})$. $CR(\mathbf{A}) \geq 0$ and $CR(\mathbf{A}) = 0$ if and only if $\mathbf{A}$ is consistent. Saaty suggested the rule of acceptability $CR < 0.1$.

Inefficiency

Example of Blanquero, Carrizosa and Conde (2006, p. 282):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 2 \\ 1/2 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/2 & 1/3 & 2 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1 \\ 2.0712416 \end{pmatrix}, \quad \mathbf{w}^{*} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1.003 \\ 2.0712416 \end{pmatrix}.$$

i	a_{i3}	x_{i3}^{EM}	x_{i3}^{*}	$\lvert a_{i3} - x_{i3}^{EM}\rvert$	$\lvert a_{i3} - x_{i3}^{*}\rvert$
1	6	6.01438057	5.99639139	0.01438057	0.00360859
2	4	4.26049429	4.24775103	0.26049429	0.24775103
3	1	1	1	0	0
4	2	2.07124160	2.06504646	0.07124160	0.06504646

(Internal) inefficiency

Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathcal{PCM}_n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ be a positive weight vector.

Definition. $\mathbf{w}$ is called *inefficient* if there exists a weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^T$ such that $|a_{ij} - w'_i/w'_j| \leq |a_{ij} - w_i/w_j|$ for all i, j, and there exist k, ℓ such that $|a_{k\ell} - w'_k/w'_\ell| < |a_{k\ell} - w_k/w_\ell|$.

Definition. $\mathbf{w}$ is called *internally inefficient* if there exists a weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^T$ such that $a_{ij} \leq w'_i/w'_j \leq w_i/w_j$ if $a_{ij} \leq w_i/w_j$, and $a_{ij} \geq w'_i/w'_j \geq w_i/w_j$ if $a_{ij} \geq w_i/w_j$ are fulfilled for all i, j, and there exist k, ℓ such that $w'_k/w'_\ell < w_k/w_\ell$ if $a_{k\ell} \leq w_k/w_\ell$, and $w'_k/w'_\ell > w_k/w_\ell$ if $a_{k\ell} \geq w_k/w_\ell$.

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 1/9 & 9 & 1/9 & 1/8 \\ 1/4 & 1 & 1/8 & 1/4 & 1/7 & 1/5 \\ 9 & 8 & 1 & 8 & 4 & 1/2 \\ 1/9 & 4 & 1/8 & 1 & 7 & 1/3 \\ 9 & 7 & 1/4 & 1/7 & 1 & 1/5 \\ 8 & 5 & 2 & 3 & 5 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.1281 \\ 0.0180 \\ 0.3028 \\ 0.1237 \\ 0.1440 \\ 0.2835 \end{pmatrix}, \mathbf{w}^* = \begin{pmatrix} 0.1281 \\ \mathbf{0.0206} \\ \mathbf{0.3471} \\ 0.1237 \\ 0.1440 \\ \mathbf{0.3249} \end{pmatrix}.$$

Approximations are

$$\mathbf{X}^{EM} \qquad\qquad \mathbf{X}^*$$

$$\begin{pmatrix} 1 & 7.13 & 0.42 & 1.03 & 0.88 & 0.45 \\ 0.14 & 1 & 0.05 & 0.14 & 0.12 & 0.06 \\ 2.36 & 16.86 & 1 & 2.44 & 2.10 & 1.06 \\ 0.96 & 6.88 & 0.40 & 1 & 0.85 & 0.43 \\ 1.12 & 8.02 & 0.47 & 1.16 & 1 & 0.50 \\ 2.21 & 15.78 & 0.93 & 2.29 & 1.96 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{6.22} & \mathbf{0.36} & 1.03 & 0.88 & \mathbf{0.39} \\ \mathbf{0.16} & 1 & 0.05 & \mathbf{0.16} & \mathbf{0.14} & 0.06 \\ \mathbf{2.71} & 16.86 & 1 & \mathbf{2.80} & \mathbf{2.40} & 1.06 \\ 0.96 & \mathbf{6.01} & \mathbf{0.35} & 1 & 0.85 & \mathbf{0.38} \\ 1.12 & \mathbf{7.00} & \mathbf{0.41} & 1.16 & 1 & \mathbf{0.44} \\ \mathbf{2.53} & 15.78 & 0.93 & \mathbf{2.62} & \mathbf{2.25} & 1 \end{pmatrix}$$

$$\mathbf{A}(p,q) = \begin{pmatrix} 1 & p & p & p & \dots & p & p \\ 1/p & 1 & q & 1 & \dots & 1 & 1/q \\ 1/p & 1/q & 1 & q & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 1/p & 1 & 1 & 1 & \dots & 1 & q \\ 1/p & q & 1 & 1 & \dots & 1/q & 1 \end{pmatrix},$$

Proposition. Let q be positive and $q \neq 1$. Then $\mathbf{w}^{EM}$ is internally inefficient, therefore inefficient. Furthermore,CR inconsistency can be arbitrarily small if q is close enough to 1.

Sketch of the proof. If $p > 1$, then with $\mathbf{w}^* = (p, 1, \dots, 1)^T$ we have $x_{1j}^{EM} < x^*_{1j} = p \ (j = 2, 3, \dots, n)$.

Eigenvector method as the solution of optimization problems:

- Fichtner' metric
- $\min\max$ and $\max\min$ problems of Perron and Frobenius

Fichtner' metric

Theorem (Fichtner, 1984) Let $\delta : \mathcal{PCM}_n \times \mathcal{PCM}_n \to \mathbb{R}$ be as follows:

$$\delta(\mathbf{A},\mathbf{B}) \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n}\left(w_i^{EM(\mathbf{A})} - w_i^{EM(\mathbf{B})}\right)^2} + \frac{|\lambda_{\max}(\mathbf{A}) - \lambda_{\max}(\mathbf{B})|}{2(n-1)} + $$

$$+\chi(\mathbf{A},\mathbf{B})\,\frac{|\lambda_{\max}(\mathbf{A}) + \lambda_{\max}(\mathbf{B}) - 2n|}{2(n-1)},$$

where

$$\chi(\mathbf{A},\mathbf{B}) = \begin{cases} 0 & \text{if } \mathbf{A} = \mathbf{B}, \\ 1 & \text{if } \mathbf{A} \neq \mathbf{B}. \end{cases}$$

Then, δ is a metric in $\mathcal{PCM}_n$ with the following properties:

Fichtner' metric

(a) for every $\mathbf{A} \in \mathcal{PCM}_n$, $\mathbf{X}^{EM(\mathbf{A})}$ is the optimal solution of the problem $\min\{\delta(\mathbf{A},\mathbf{X}) | \mathbf{X} \text{ is consistent}\}$;

(b)
$\min\{\delta(\mathbf{A},\mathbf{X}) | \mathbf{X} \text{ is consistent}\} = \delta(\mathbf{A},\mathbf{X}^{EM(\mathbf{A})}) = \frac{\lambda_{\max}(\mathbf{A})-n}{n-1}$.

Note that Fichtner's metric is not continuous.

Theorem (Perron, Frobenius). Let $\mathbf{A} \in \mathcal{PCM}_n$, and the largest eigenvalue of $\mathbf{A}$ be denoted by $\lambda_{\max}$. Then

$$\max_{\mathbf{w}\in\mathbb{R}^n_+} \min_{1\leq i\leq n} \frac{\sum\limits_{j=1}^{n} a_{ij}w_j}{w_i} \leq \lambda_{\max} \leq \min_{1\leq i\leq n} \max_{\mathbf{w}\in\mathbb{R}^n_+} \frac{\sum\limits_{j=1}^{n} a_{ij}w_j}{w_i}$$

where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. Furthermore, both inequalities hold with equality if and only if $\mathbf{w} = \kappa\mathbf{w}^{EM}$, where κ is an arbitrary positive number.

Conclusion. Optimality with respect to reasonable and nice objective functions does not exclude inefficiency.

Main references

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Thank you for attention.

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