The logarithmic least squares optimality of the geometric mean of weight vectors calculated from all spanning trees for (in)complete pairwise comparison matrices

Sándor Bozóki

Institute for Computer Science and Control
Hungarian Academy of Sciences (MTA SZTAKI);

Corvinus University of Budapest

Vitaliy Tsyganok

Laboratory for Decision Support Systems,
The Institute for Information Recording of National Academy of Sciences of Ukraine;

Department of System Analysis, State University of Telecommunications

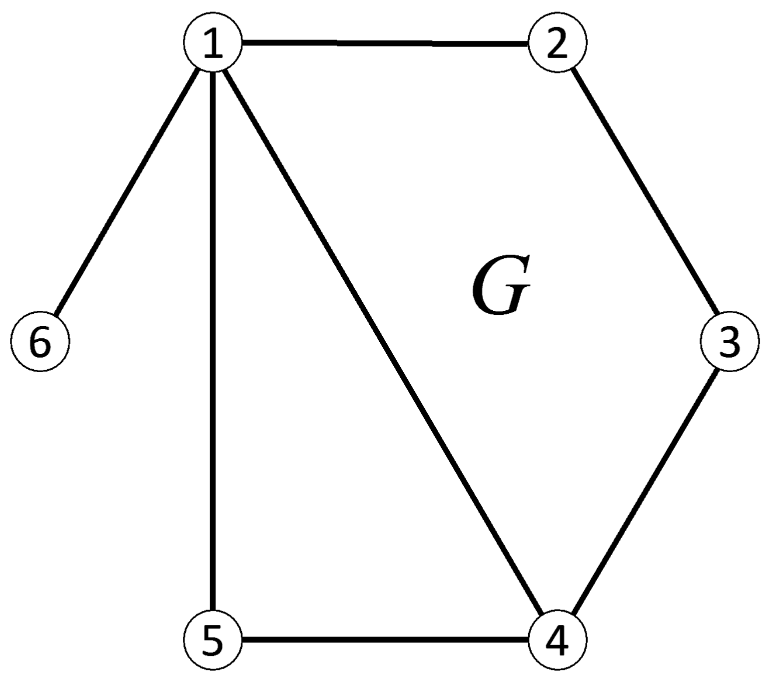
MCDM, Ottawa
July 12, 2017

incomplete pairwise comparison matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$

incomplete pairwise comparison matrix and its graph

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



The Logarithmic Least Squares (LLS) problem

$$\min \sum_{\substack{i,j: \\ a_{ij} \text{ is known}}} \left[\log a_{ij} - \log\left(\frac{w_i}{w_j}\right)\right]^2$$

$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

The most common normalizations are $\sum_{i=1}^{n} w_i = 1$, $\prod_{i=1}^{n} w_i = 1$ and $w_1 = 1$.

Theorem (Bozóki, Fülöp, Rónyai, 2010): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph G is connected. Then the optimal solution $\mathbf{w} = \exp \mathbf{y}$ of the logarithmic least squares problem is the unique solution of the following system of linear equations:

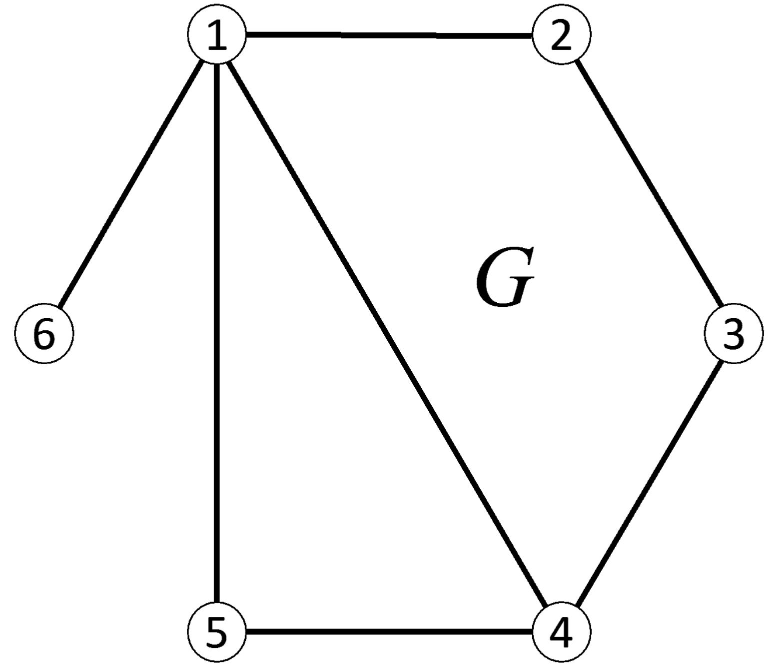
$$(\mathbf{L}\mathbf{y})_i = \sum_{k: e(i,k) \in E(G)} \log a_{ik} \qquad \text{for all } i = 1, 2, \ldots, n,$$

$$y_1 = 0$$

where $\mathbf{L}$ denotes the Laplacian matrix of G (ℓ_{ii} is the degree of node i and $\ell_{ij} = -1$ if nodes i and j are adjacent).

example

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1(=0) \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} \log(a_{12}\, a_{14}\, a_{15}\, a_{16}) \\ \log(a_{21}\, a_{23}) \\ \log(a_{32}\, a_{34}) \\ \log(a_{41}\, a_{43}\, a_{45}) \\ \log(a_{51}\, a_{54}) \\ \log a_{61} \end{pmatrix}$$

Pairwise Comparison Matrix Calculator (PCMC)

The logarithmic least squares optimal weight vector can be calculated at

pcmc.online

CR-minimal ($\lambda_{\max}$-minimal) completion is also calculated.

PCMC deals with Pareto optimality (efficiency) of weight vectors, too.

Pareto optimality (efficiency)

Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$ be an $n \times n$ pairwise comparison matrix and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector.

Definition: weight vector $\mathbf{w}$ is called *efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n,$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \leq k, \ell \leq n.$$

Remark: A weight vector $\mathbf{w}$ is efficient if and only if $c\mathbf{w}$ is efficient , where $c > 0$ is an arbitrary scalar.

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix},$$

$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}, \quad \mathbf{w}^{*} = \begin{pmatrix} \mathbf{0.436173} \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}$$

$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}, \quad \mathbf{w}^* = \begin{pmatrix} \mathbf{0.436173} \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}$$

$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

$$\left[\frac{w_i'}{w_j'}\right] = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{3.9546} & \mathbf{8.8989} \\ \mathbf{1} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2529} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1124} & 0.1124 & 0.4444 & 1 \end{pmatrix}.$$

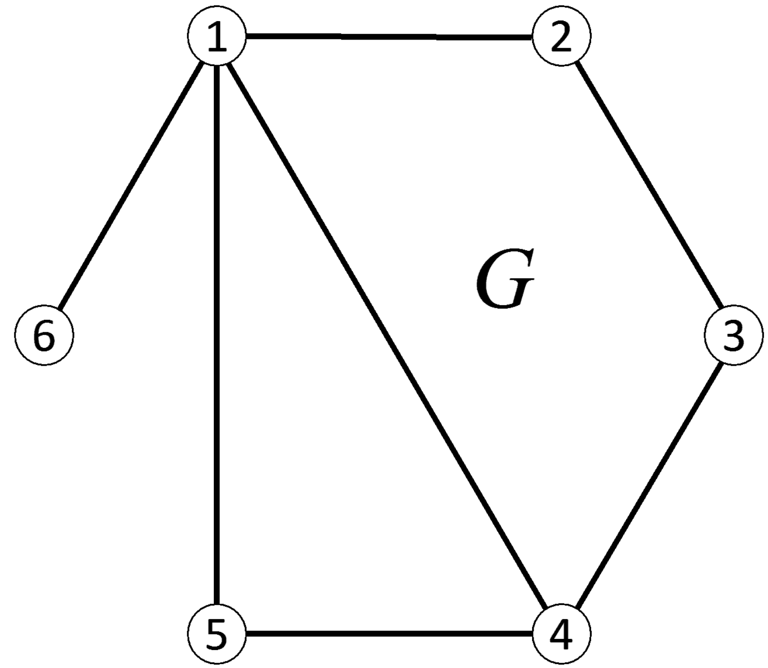
Pareto optimality (efficiency)

See more in

Bozóki, S., Fülöp, J. (2017): Efficient weight vectors from pairwise comparison matrices, European Journal of Operational Research (in print)
DOI 10.1016/j.ejor.2017.06.033

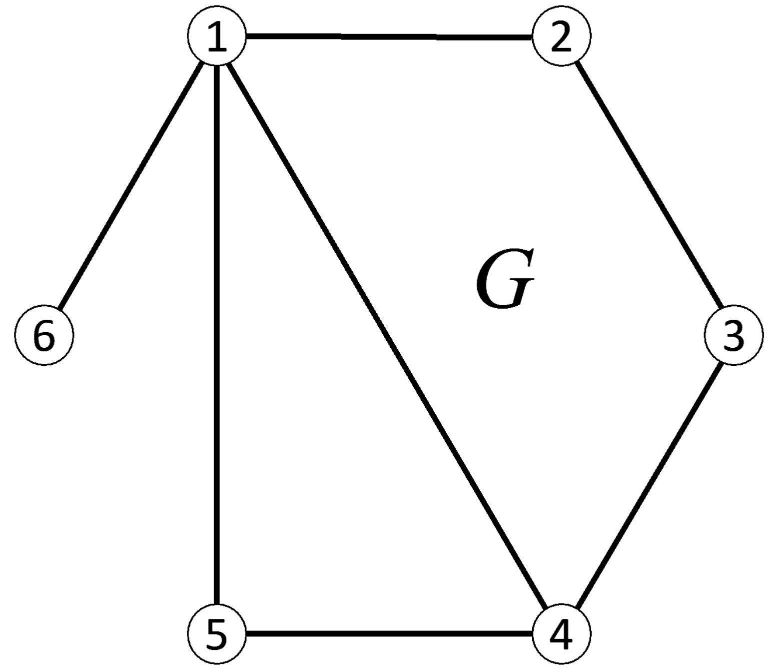
The spanning tree approach (Tsyganok, 2000, 2010)

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$

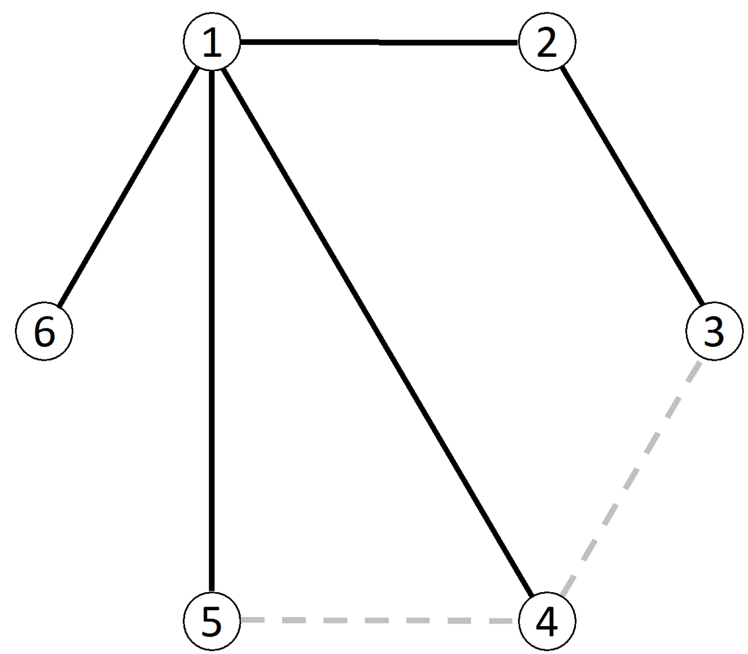


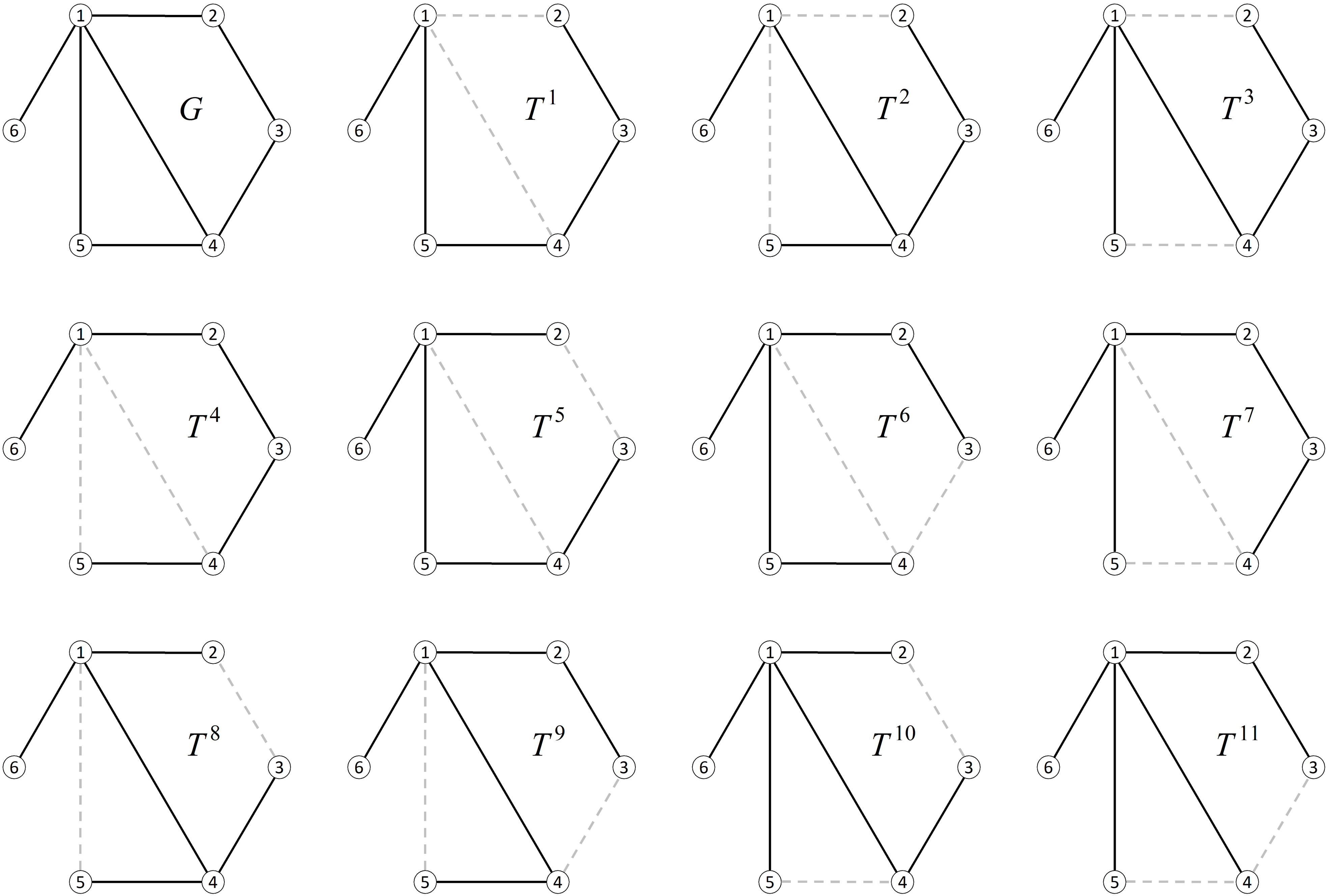
The spanning tree approach (Tsyganok, 2000, 2010)

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{34} & & \\ a_{41} & & a_{43} & 1 & a_{45} & \\ a_{51} & & & a_{54} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & & & \\ a_{41} & & & 1 & & \\ a_{51} & & & & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$




G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}
1
2
3
4
5
6

The spanning tree approach

Every spanning tree induces a weight vector.

Natural ways of aggregation: arithmetic mean, geometric mean etc.

Theorem (Lundy, Siraj, Greco, 2017): The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete pairwise comparison matrices.

Theorem (Lundy, Siraj, Greco, 2017): The geometric mean of weight vectors calculated from all spanning trees is logarithmic least squares optimal in case of complete pairwise comparison matrices.

Theorem (Bozóki, Tsyganok): Let $\mathbf{A}$ be an incomplete or complete pairwise comparison matrix such that its associated graph is connected. Then the optimal solution of the logarithmic least squares problem is equal, up to a scalar multiplier, to the geometric mean of weight vectors calculated from all spanning trees.

proof

Let G be the connected graph associated to the (in)complete pairwise comparison matrix $\mathbf{A}$ and let $E(G)$ denote the set of edges. The edge between nodes i and j is denoted by $e(i,j)$.

The Laplacian matrix of graph G is denoted by $\mathbf{L}$. Let $T^1, T^2, \ldots, T^s, \ldots, T^S$ denote the spanning trees of G, where S denotes the number of spanning trees. $E(T^s)$ denotes the set of edges in T^s.

Let $\mathbf{w}^s, s = 1, 2, \ldots, S$, denote the weight vector calculated from spanning tree T^s. Weight vector $\mathbf{w}^s$ is unique up to a scalar multiplication. Assume without loss of generality that $w_1^s = 1$.

Let $\mathbf{y}^s := \log \mathbf{w}^s, \; s = 1, 2, \ldots, S$, where the logarithm is taken element-wise.

proof

Let $\mathbf{w}^{LLS}$ denote the optimal solution to the incomplete Logarithmic Least Squares problem (normalized by $w_1^{LLS} = 1$) and $\mathbf{y}^{LLS} := \log \mathbf{w}^{LLS}$, then

$$\left(\mathbf{L}\mathbf{y}^{LLS}\right)_i = \sum_{k:e(i,k)\in E(G)} b_{ik} \qquad \text{for all } i = 1,2,\ldots,n,$$

where $b_{ik} = \log a_{ik}$ for all $e(i,k) \in E(G)$.

$b_{ik} = -b_{ki}$ for all $e(i,k) \in E(G)$.

In order to prove the theorem, it is sufficient to show that

$$\left(\mathbf{L}\frac{1}{S}\sum_{s=1}^{S}\mathbf{y}^s\right)_i = \sum_{k:e(i,k)\in E(G)} b_{ik} \qquad \text{for all } i = 1,2,\ldots,n.$$

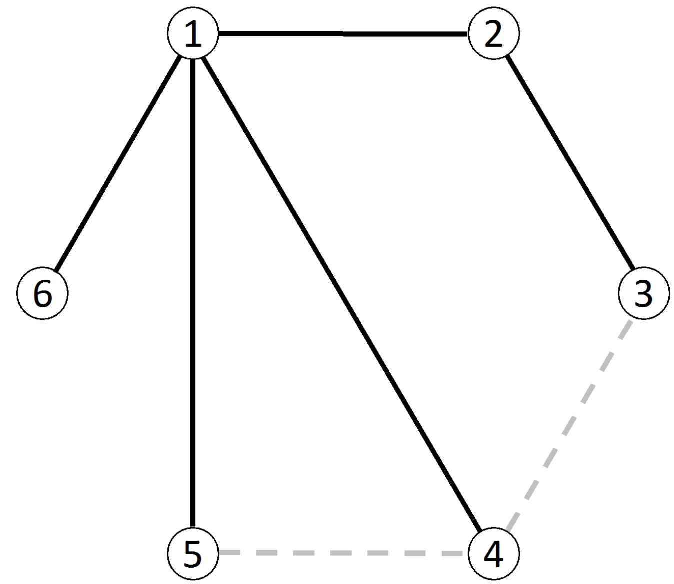
proof

Challenge: the Laplacian matrices of the spanning trees are different from the Laplacian of G.

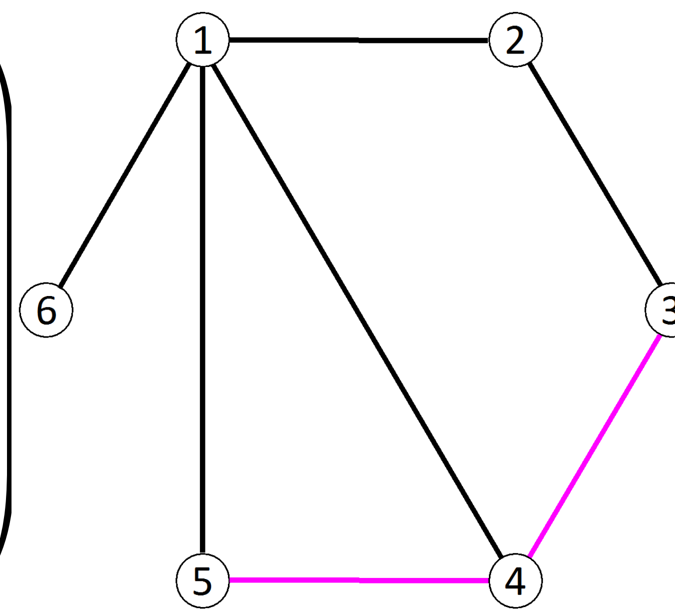
Consider an arbitrary spanning tree T^s. Then $\frac{w_i^s}{w_j^s} = a_{ij}$ for all $e(i,j) \in E(T^s)$.
Introduce the incomplete pairwise comparison matrix $\mathbf{A}^s$ by $a_{ij}^s := a_{ij}$ for all $e(i,j) \in E(T^s)$ and $a_{ij}^s := \frac{w_i^s}{w_j^s}$ for all $e(i,j) \in E(G) \backslash E(T^s)$. Again, $b_{ij}^s := \log a_{ij}^s (= y_i^s - y_j^s)$.
Note that the Laplacian matrices of $\mathbf{A}$ and $\mathbf{A}^s$ are the same ($\mathbf{L}$).

proof

$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & & & \\ a_{41} & & & 1 & & \\ a_{51} & & & & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & 1 & a_{23} & & & \\ & a_{32} & 1 & a_{32}a_{21}a_{14} & & \\ a_{41} & & a_{41}a_{12}a_{23} & 1 & a_{41}a_{15} & \\ a_{51} & & & a_{51}a_{14} & 1 & \\ a_{61} & & & & & 1 \end{pmatrix}$$



proof

Since weight vector $\mathbf{w}^s$ is generated by the matrix elements belonging to spanning tree T^s, it is the optimal solution of the LLS problem regarding $\mathbf{A}^s$, too. Equivalently, the following system of linear equations holds.

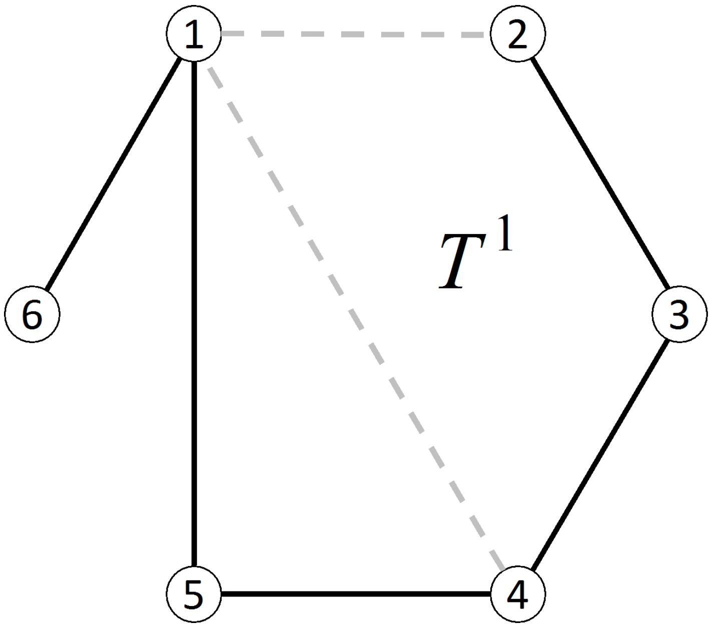
$$(\mathbf{Ly}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^s \quad \text{for all } i = 1, \ldots, n$$

proof

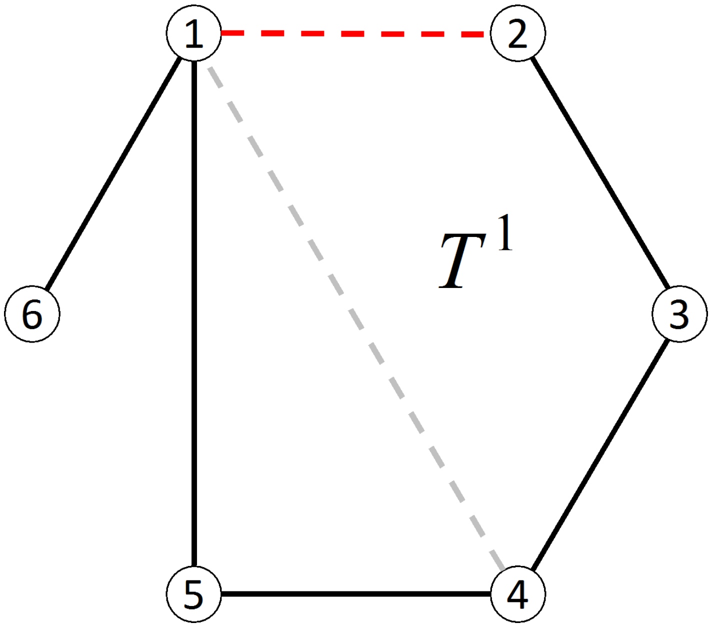
Lemma

$$\sum_{s=1}^{S}\left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b_{ik}^{s}\right) = S\sum_{k:e(i,k)\in E(G)} b_{ik}$$

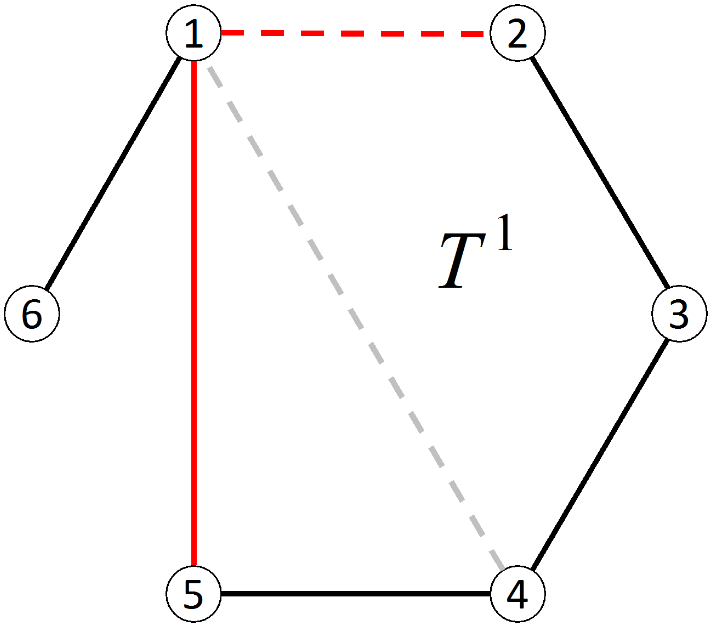
proof of the lemma



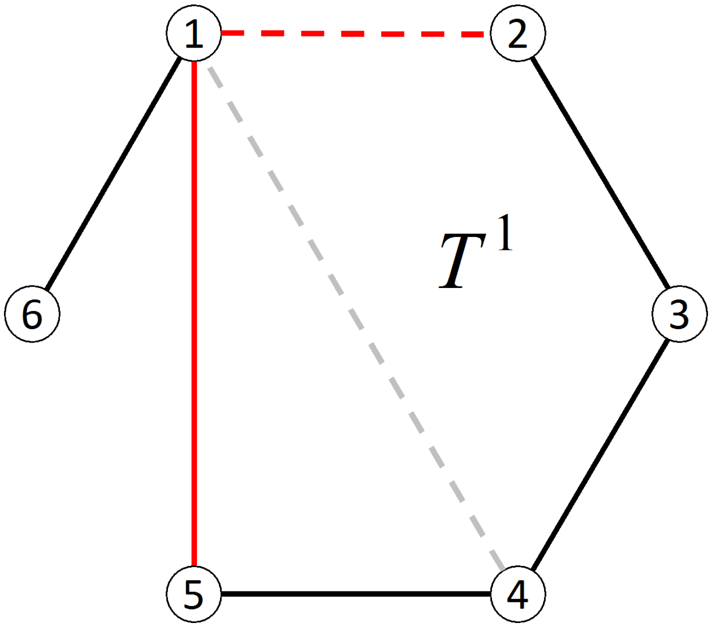
proof of the lemma



proof of the lemma

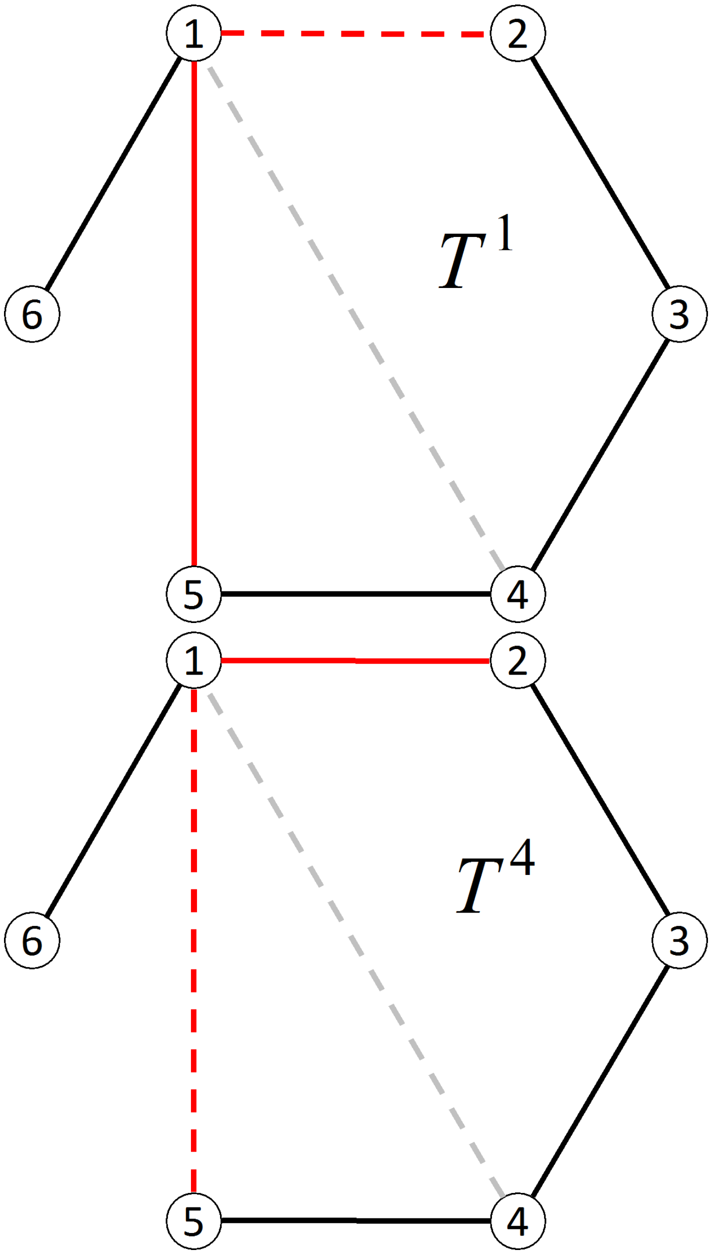


proof of the lemma



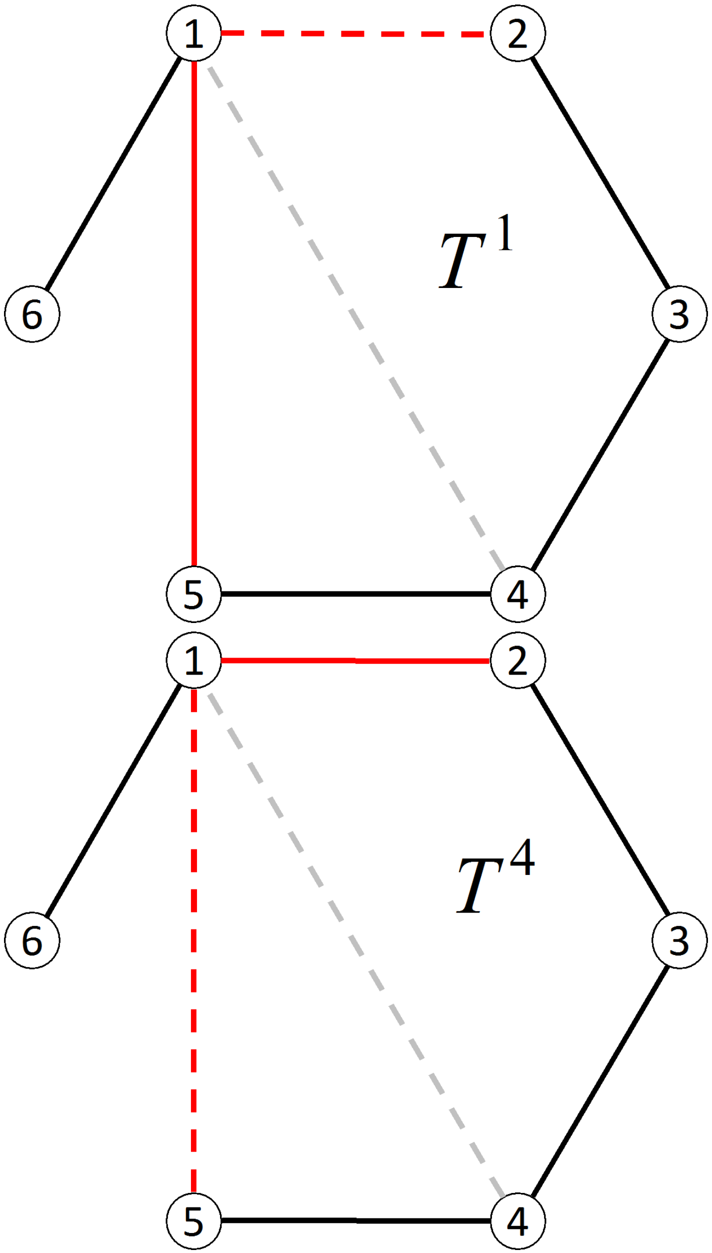
$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

proof of the lemma



$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

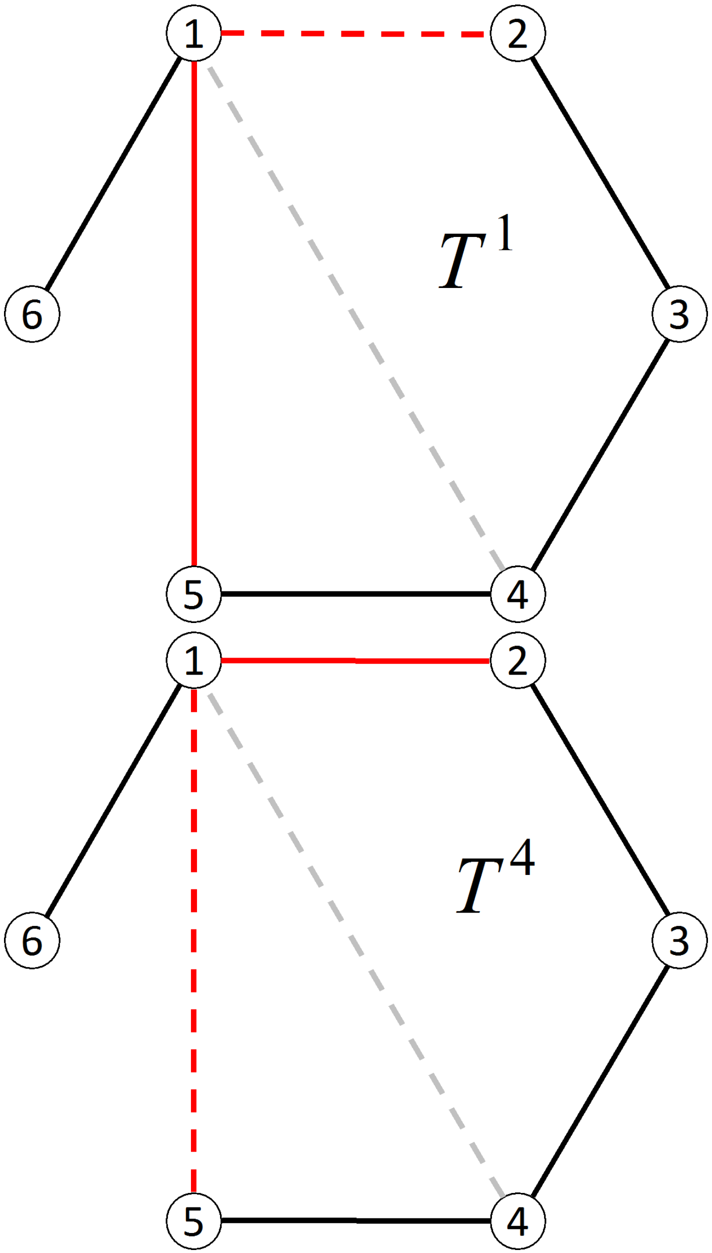
proof of the lemma



$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

proof of the lemma

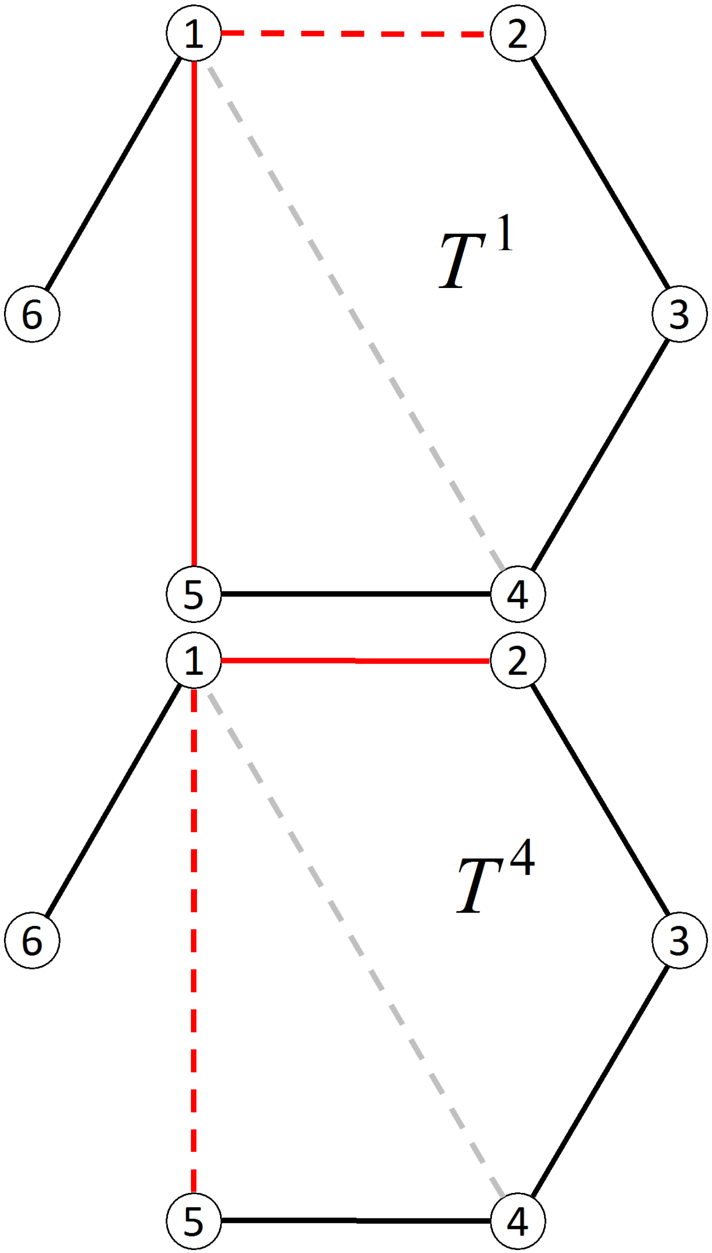


$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

---

proof of the lemma

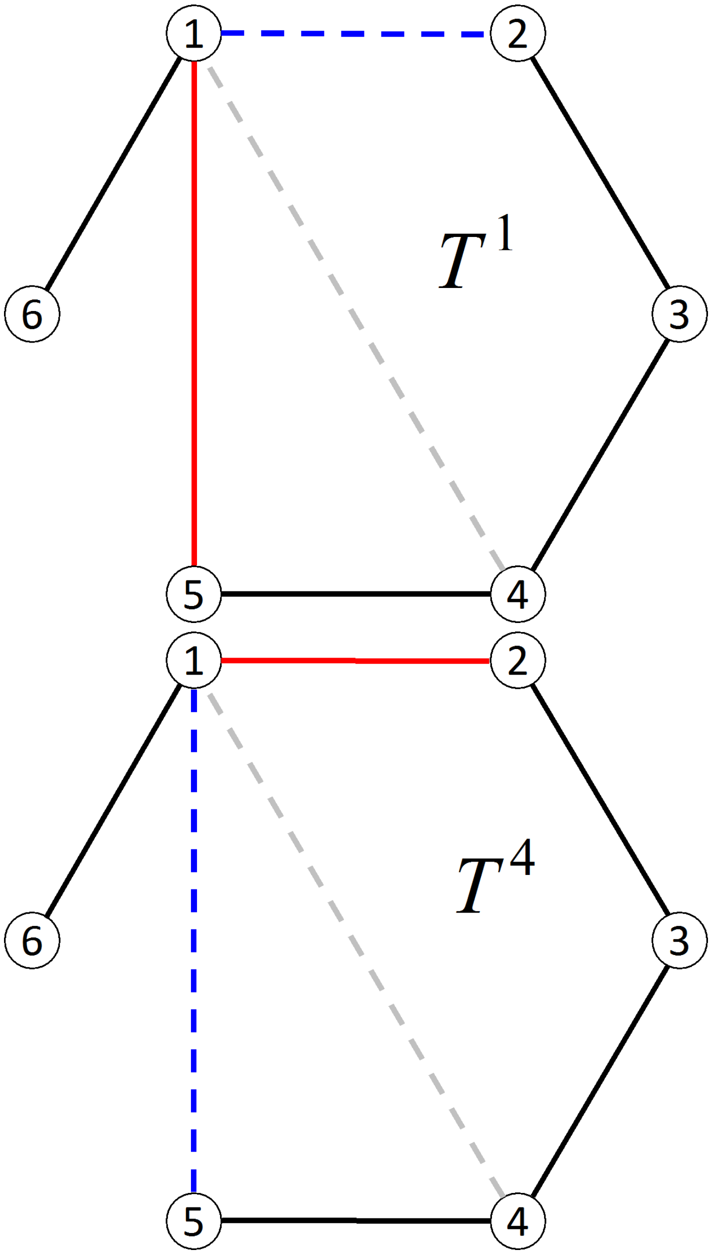


$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

$$b^1_{12} + b^4_{15} = b_{12} + b_{15}$$

proof of the lemma

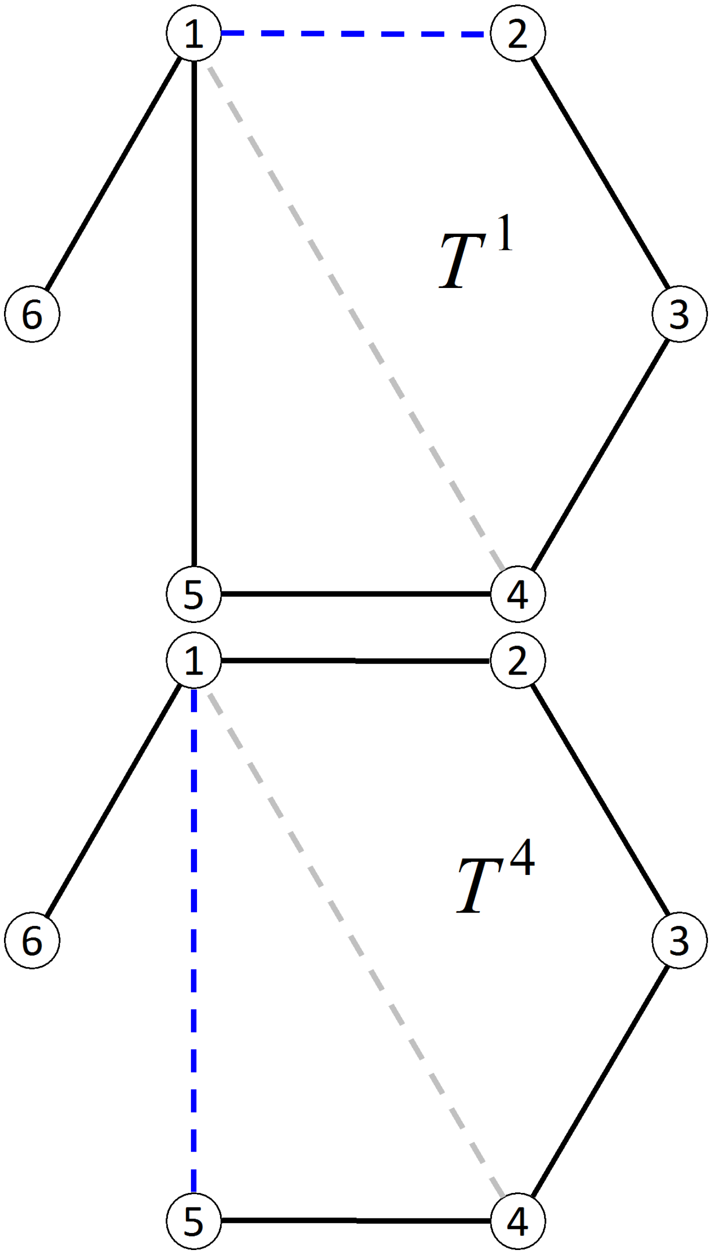


$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

$$b^1_{12} + b^4_{15} = b_{12} + b_{15}$$

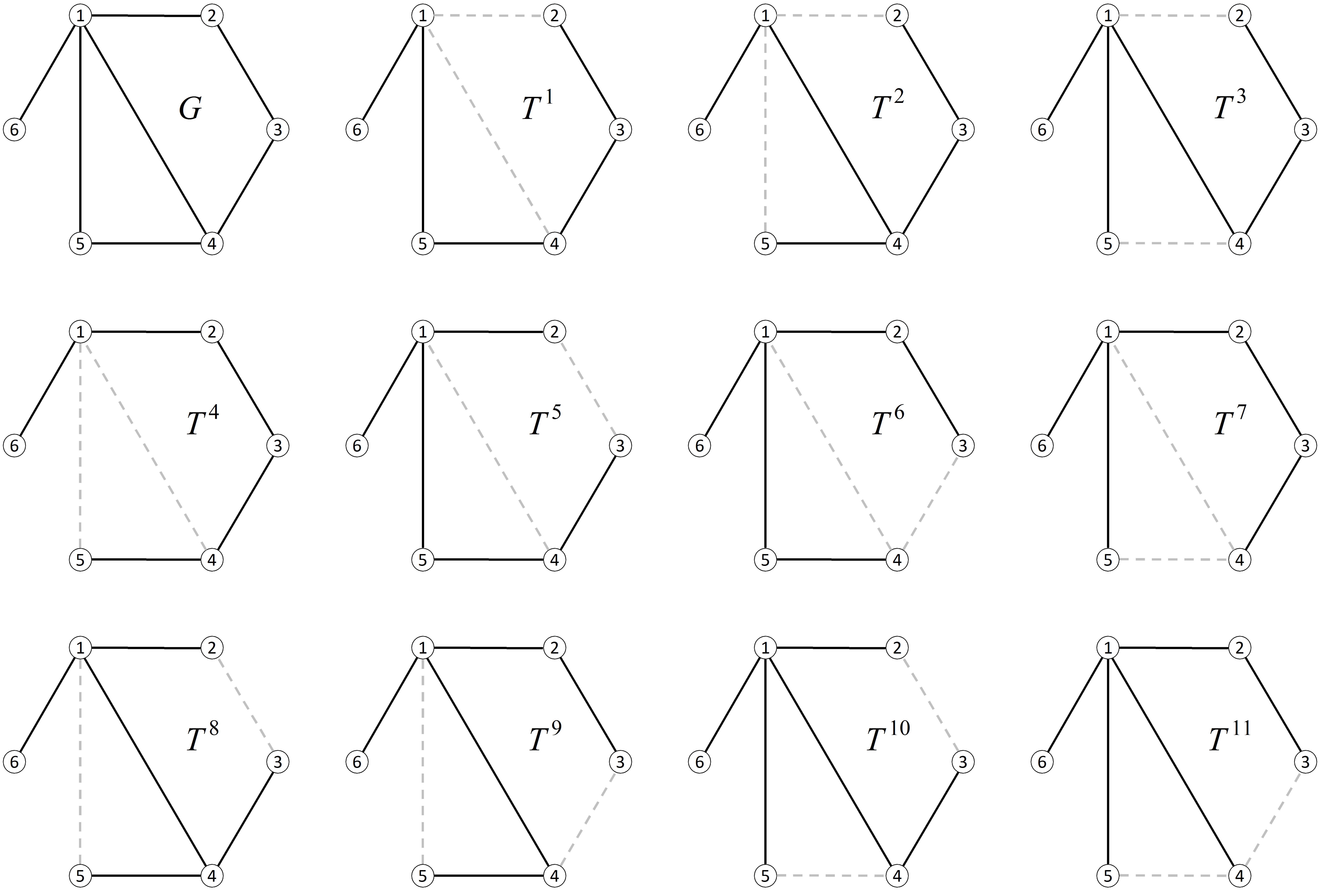
proof of the lemma

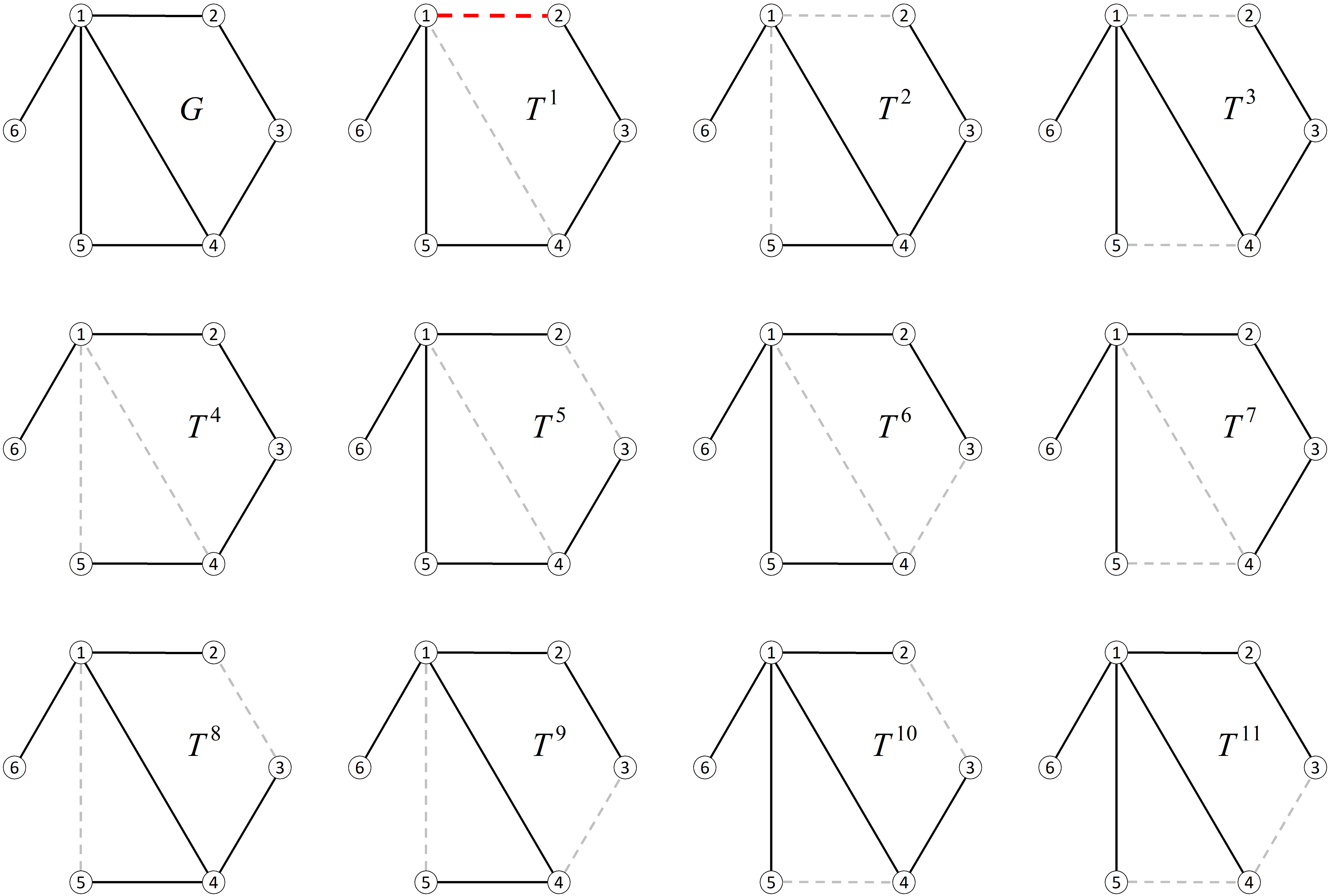


$$b^1_{12} = b_{15} + b_{54} + b_{43} + b_{32}$$

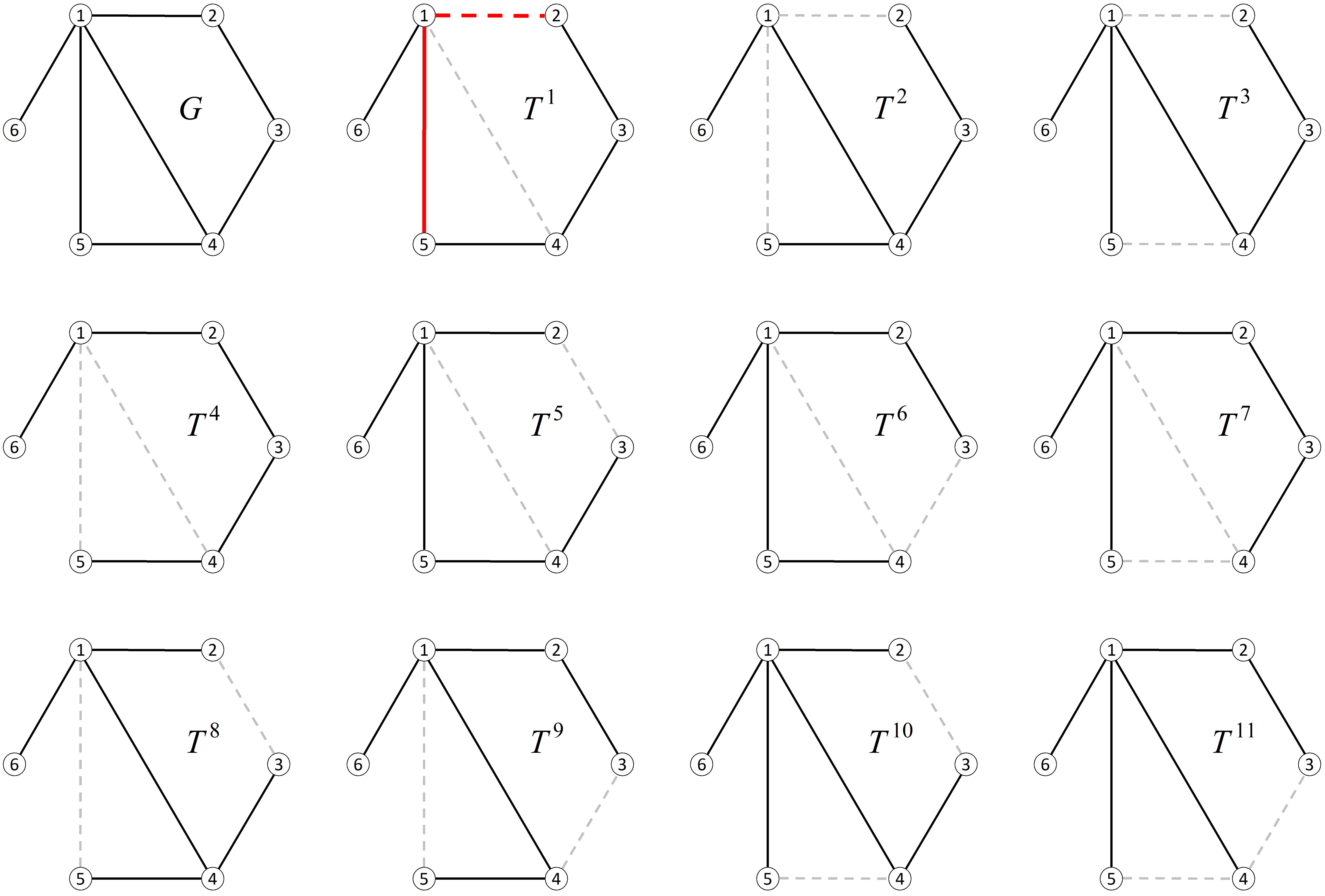
$$b^4_{15} = b_{12} + b_{23} + b_{34} + b_{45}$$

$$b^1_{12} + b^4_{15} = b_{12} + b_{15}$$

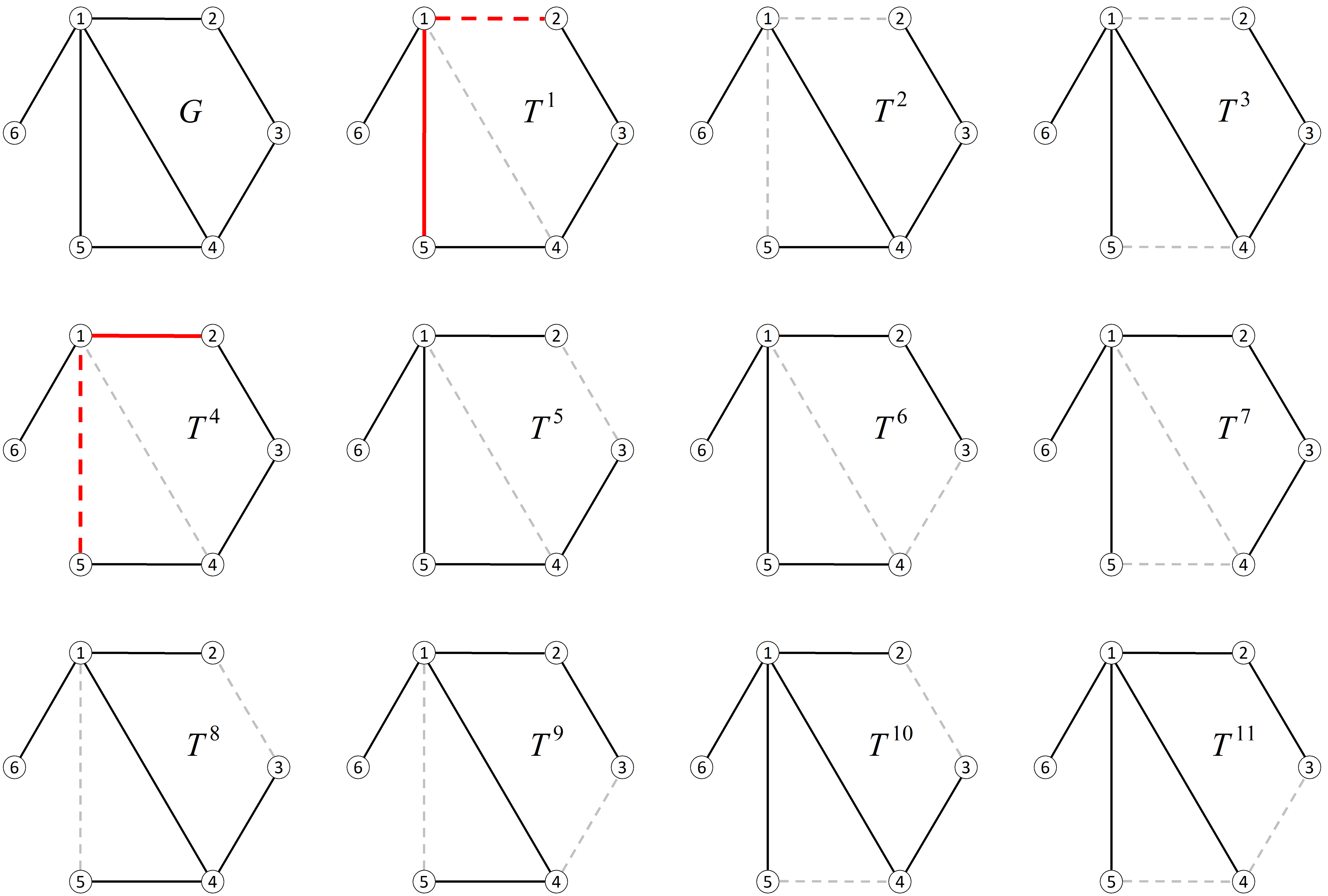

G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^10
T^11

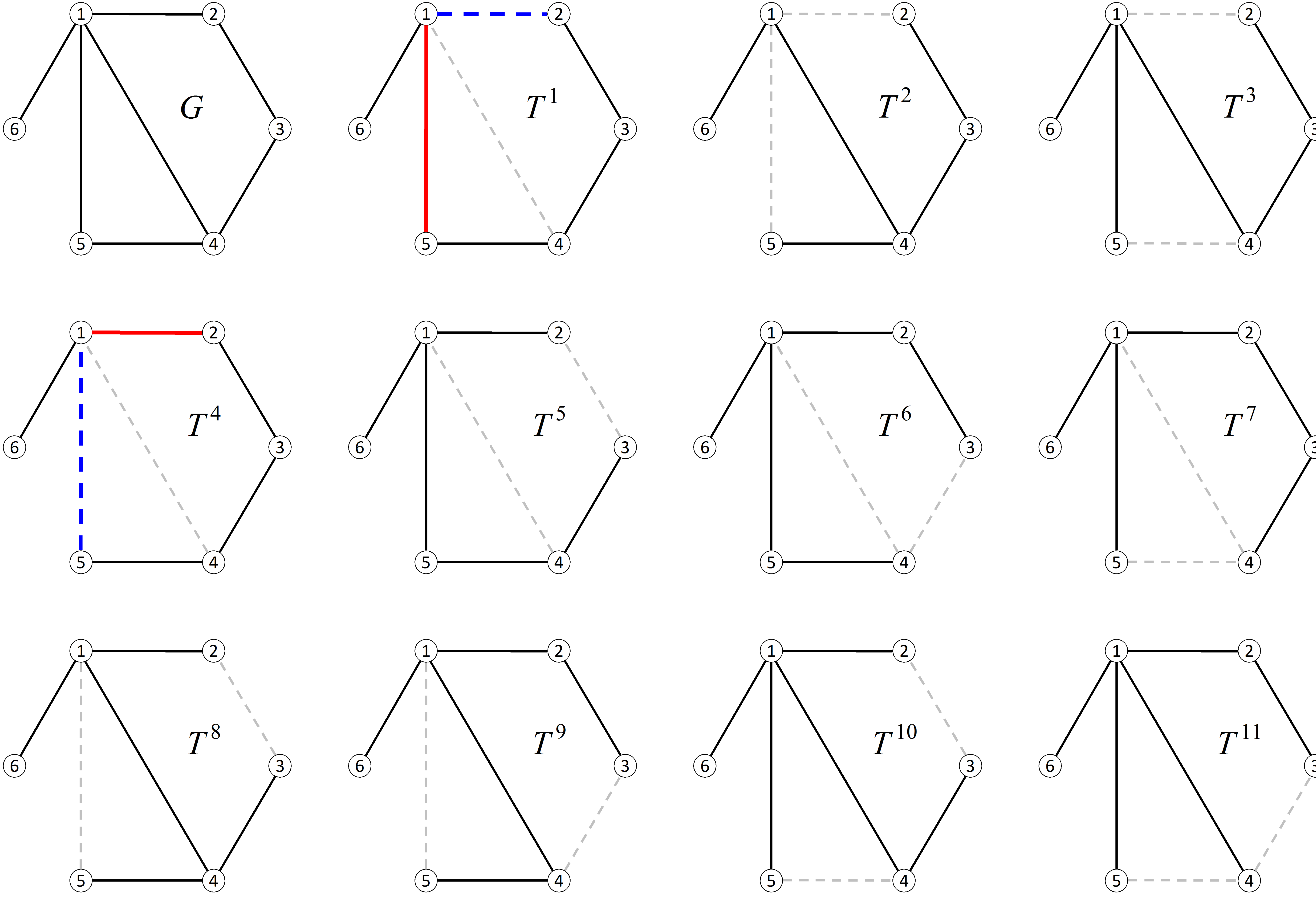


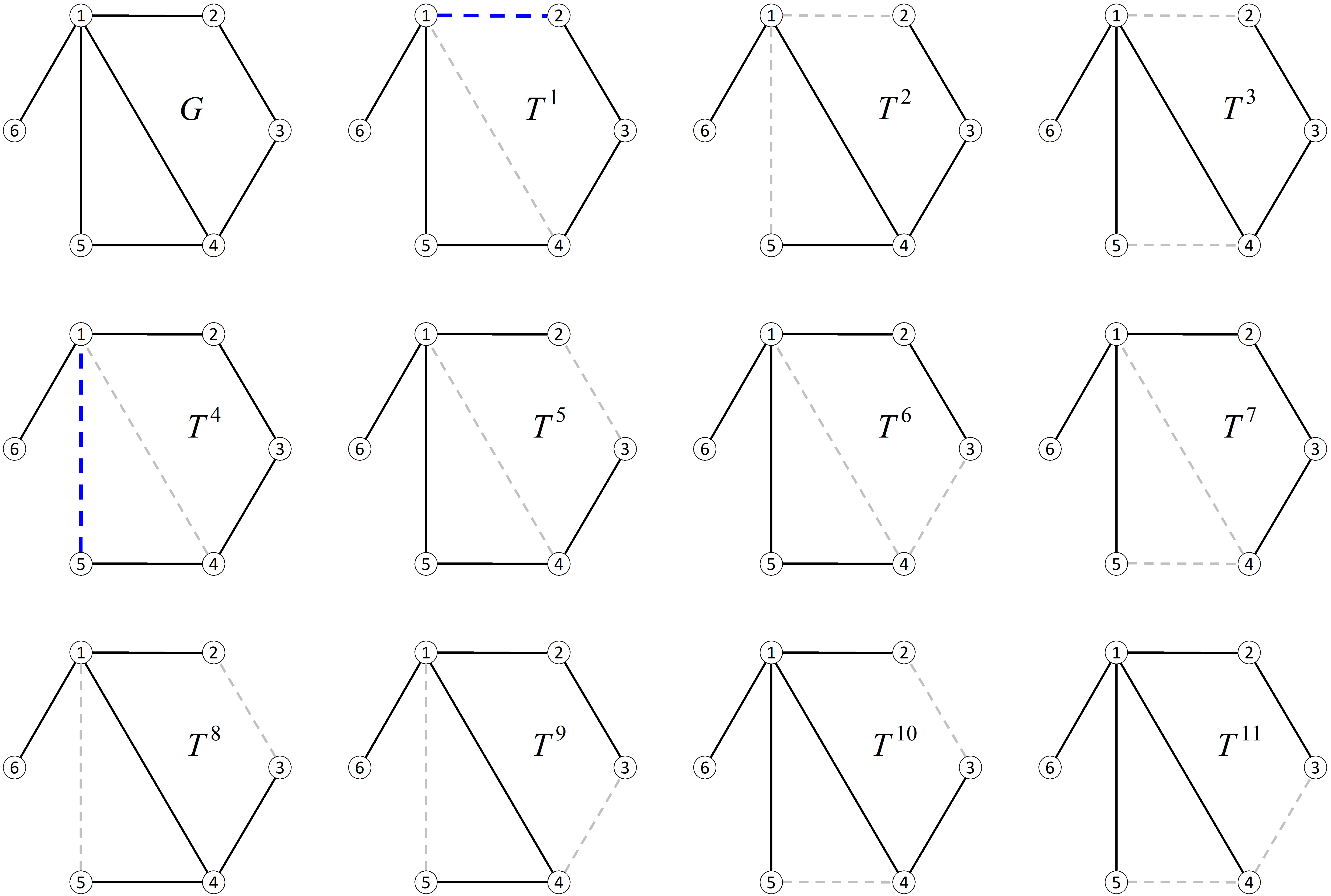
G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^10
T^11
1
2
3
4
5
6

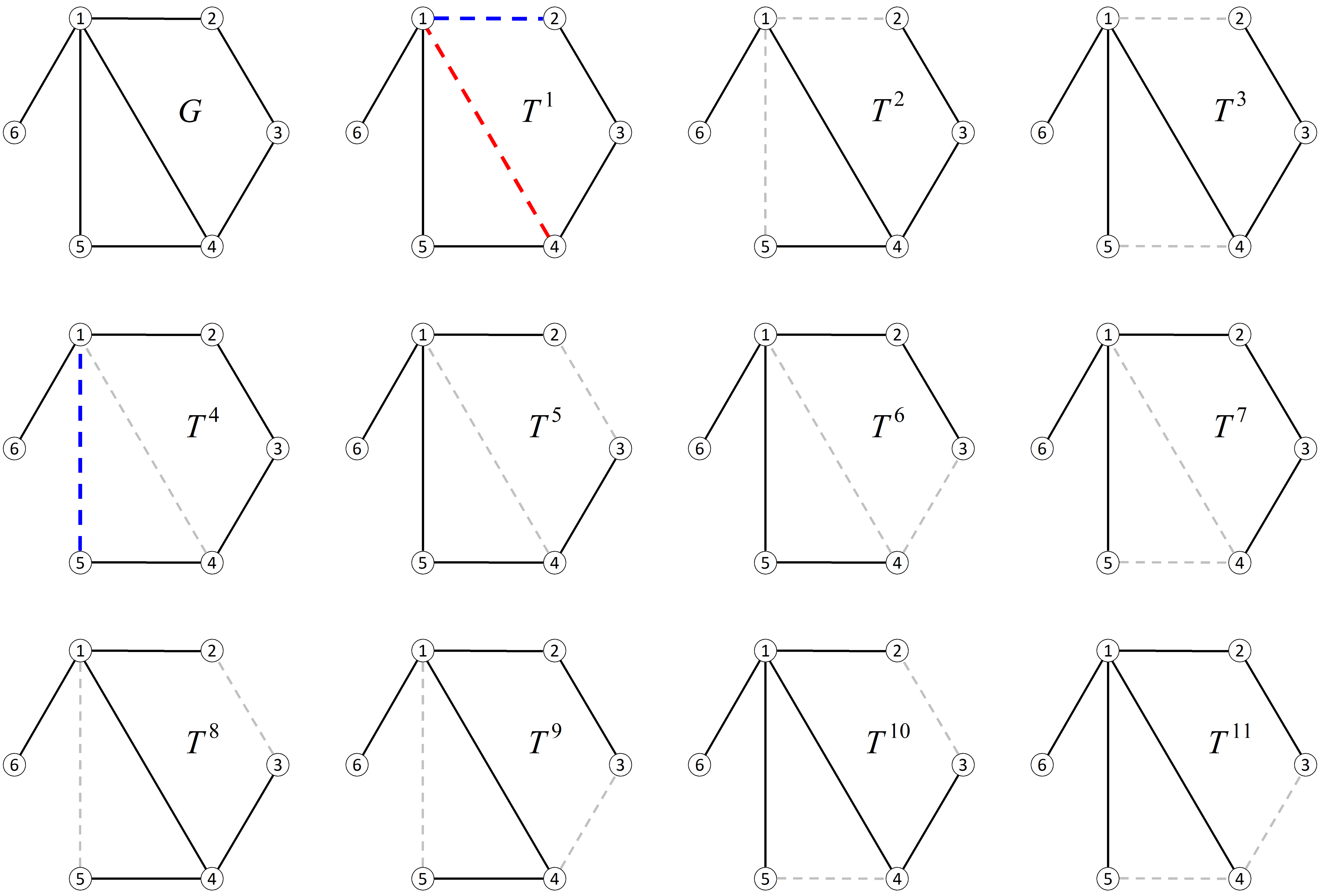


G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}

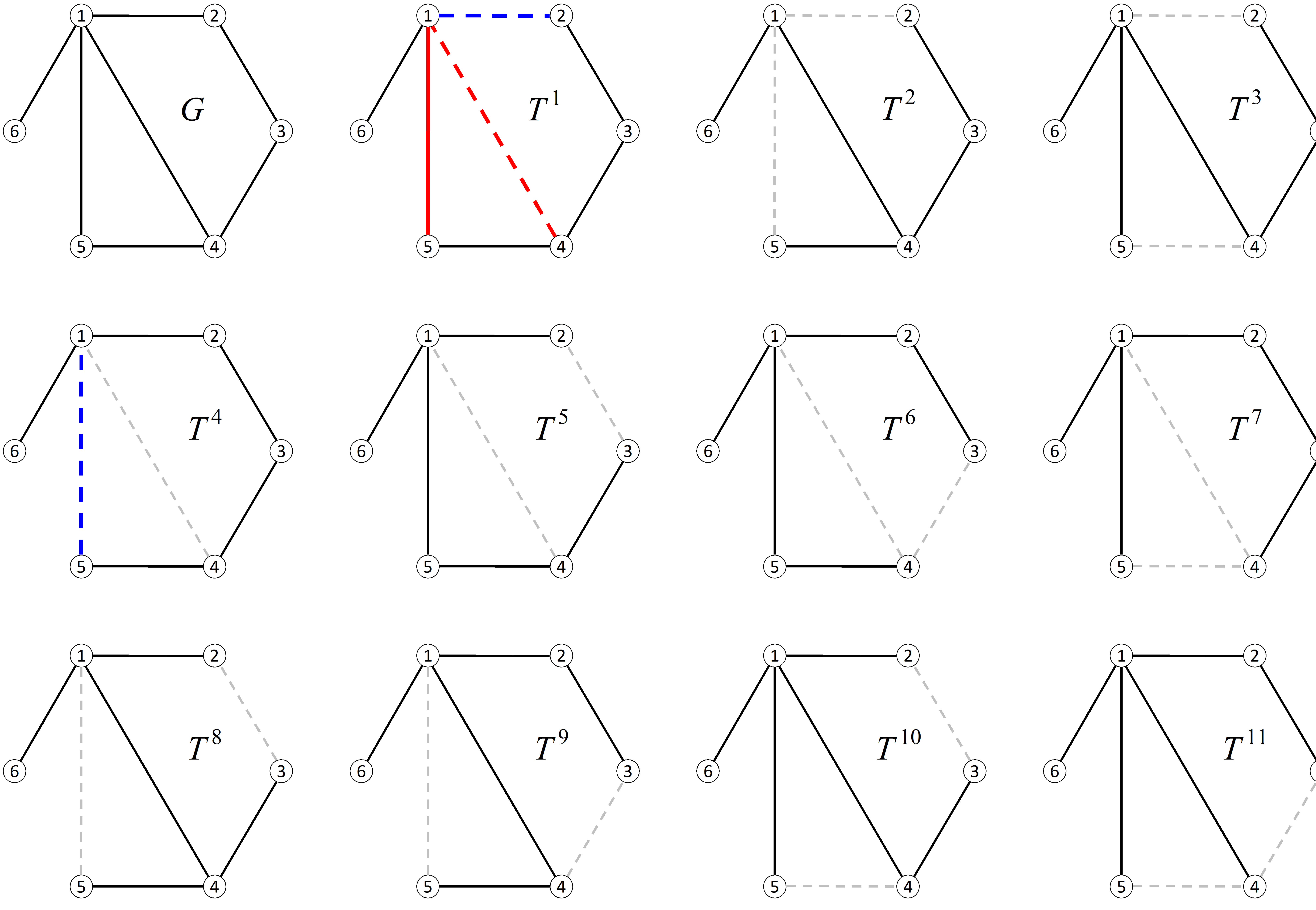

G
T 1
T 2
T 3
T 4
T 5
T 6
T 7
T 8
T 9
T 10
T 11


G
T1
T2
T3
T4
T5
T6
T7
T8
T9
T10
T11
1
2
3
4
5
6

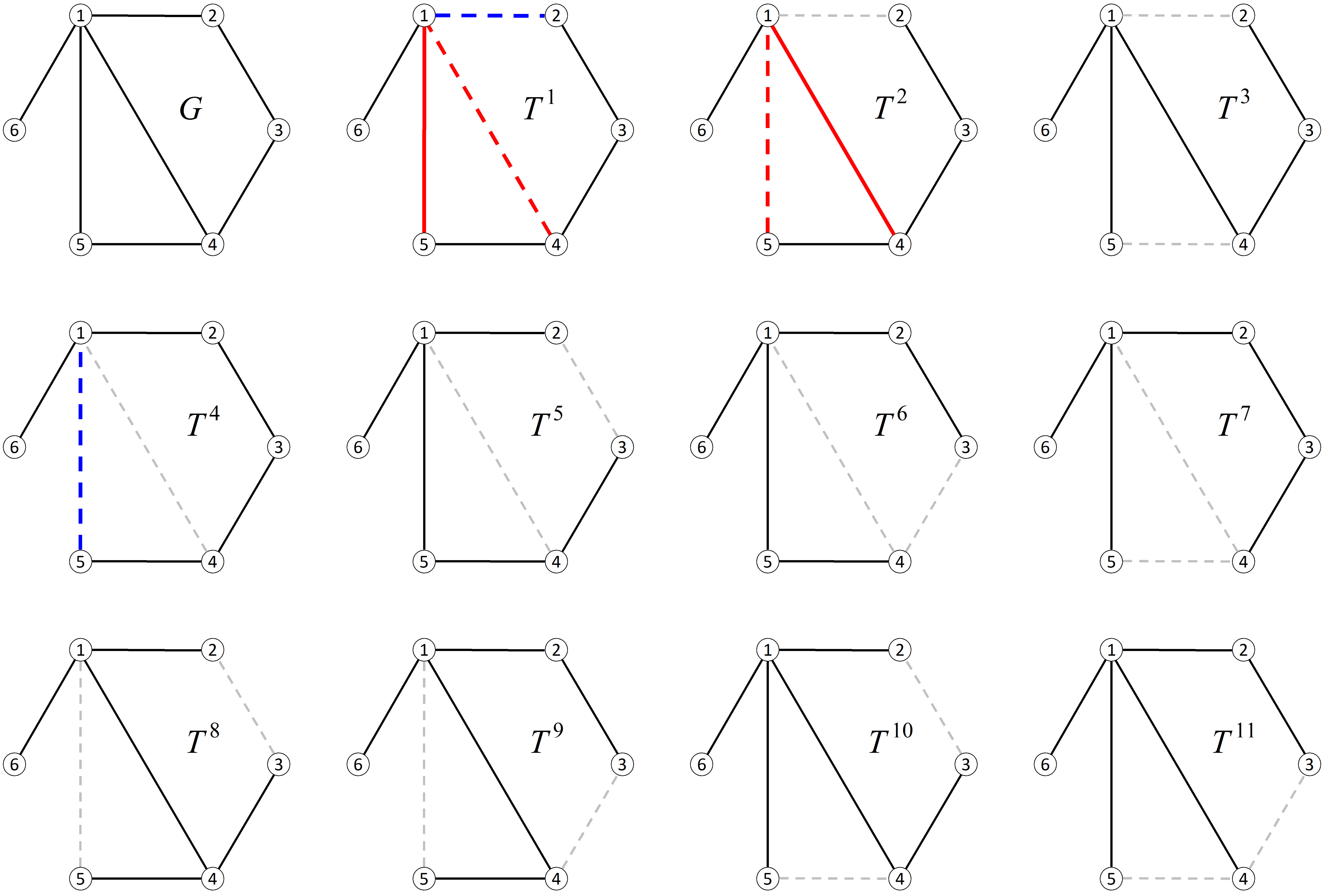

G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^10
T^11
1
2
3
4
5
6

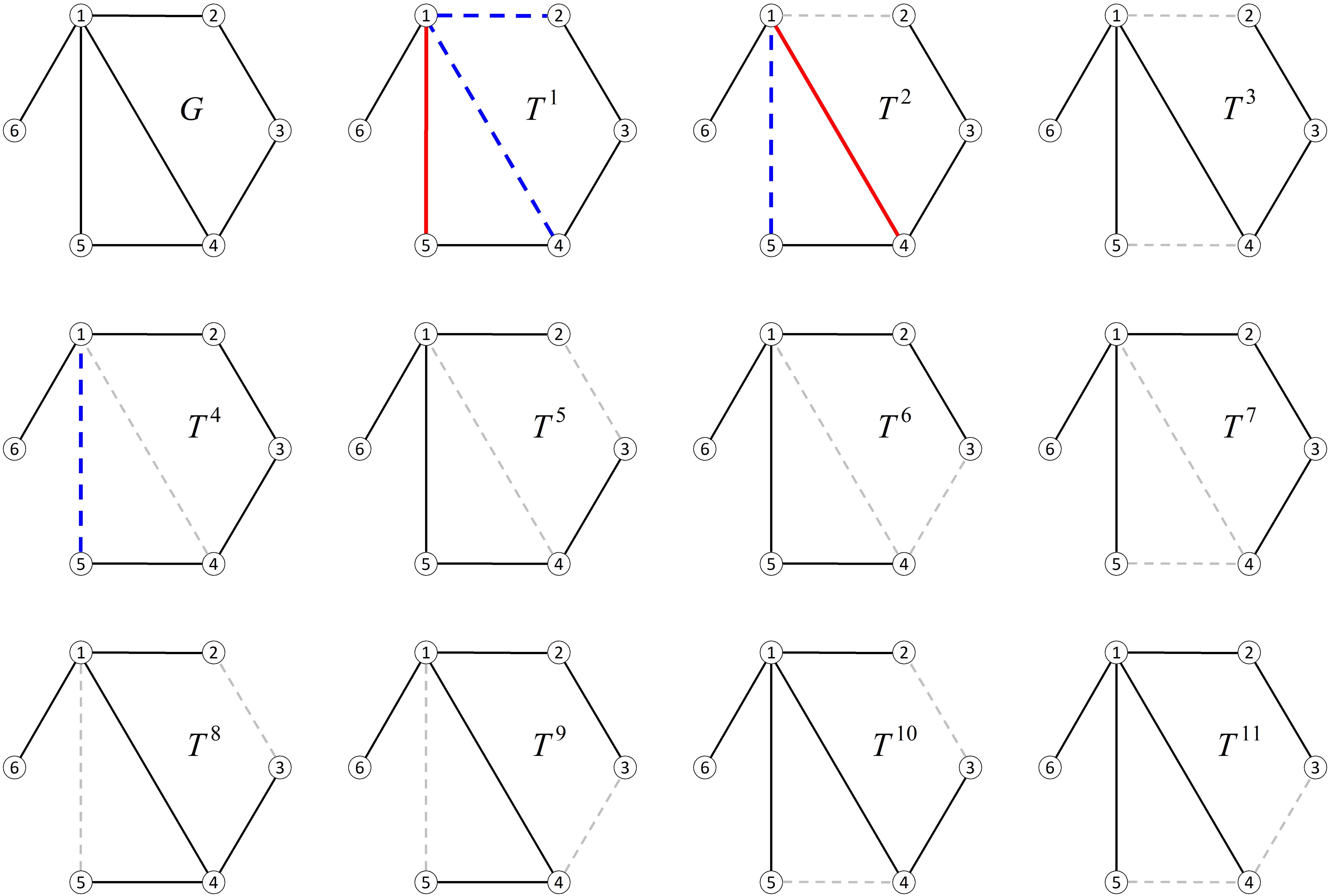


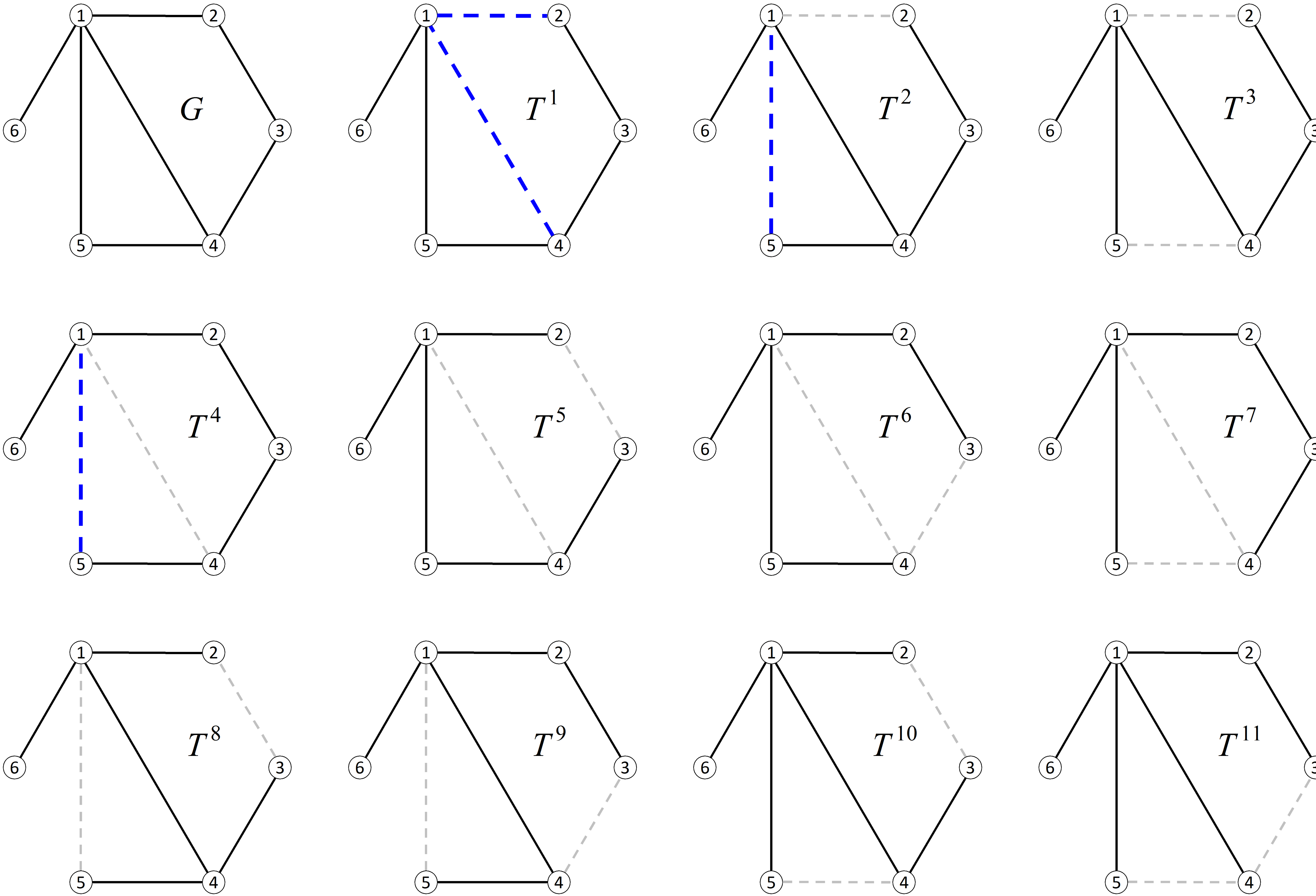
G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}
1
2
3
4
5
6

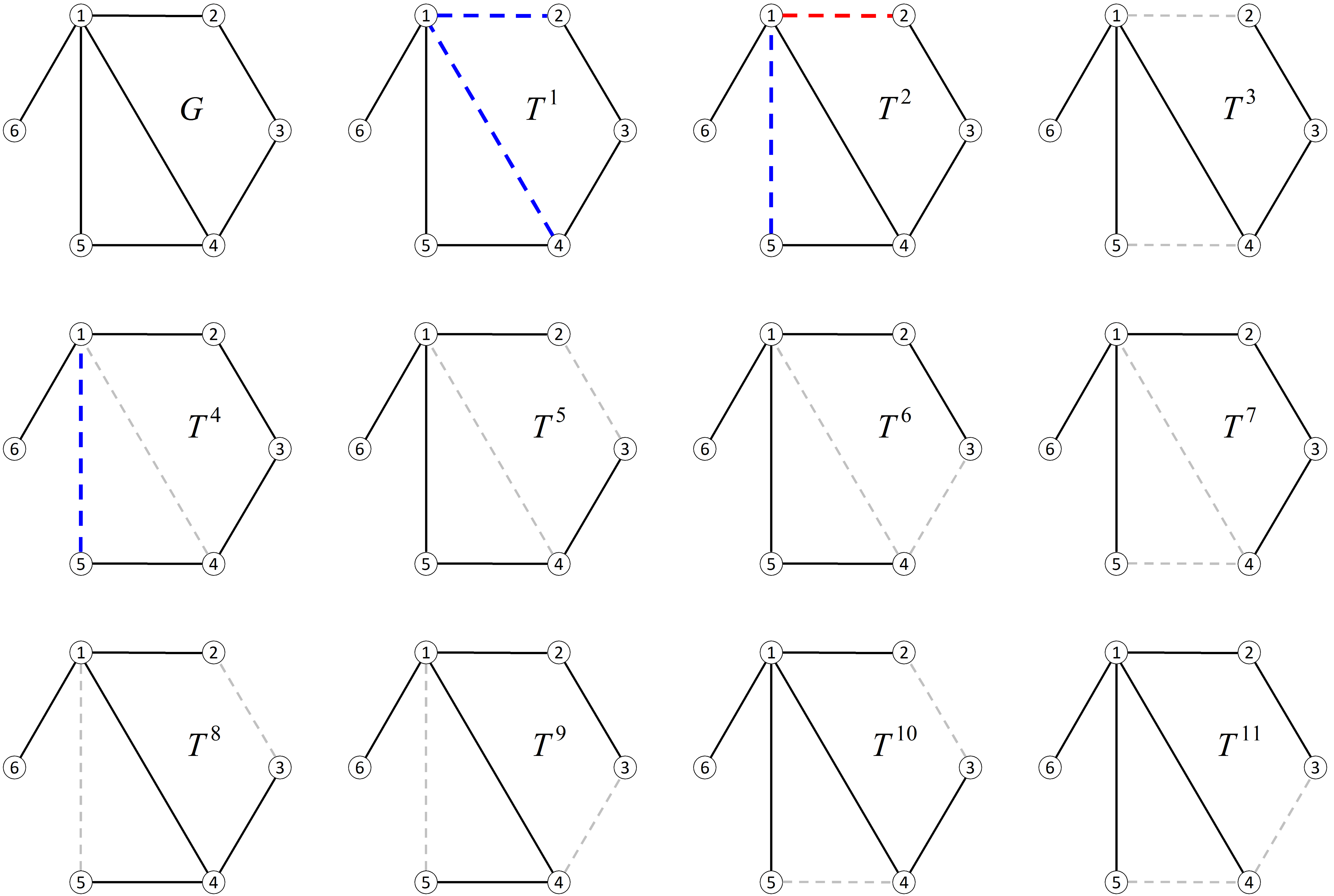


G
T 1
T 2
T 3
T 4
T 5
T 6
T 7
T 8
T 9
T 10
T 11

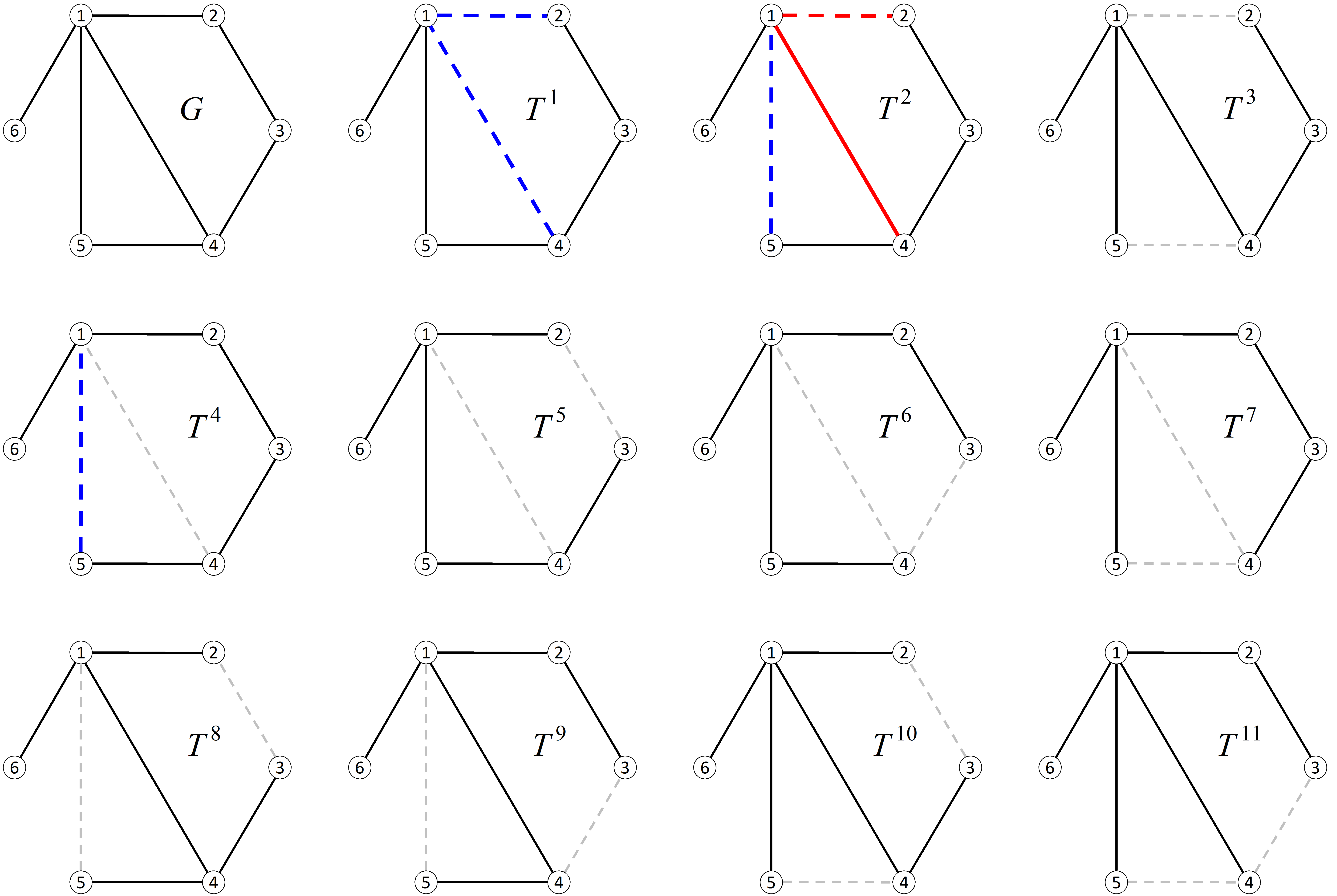

G
T 1
T 2
T 3
T 4
T 5
T 6
T 7
T 8
T 9
T 10
T 11
1
2
3
4
5
6


G
T 1
T 2
T 3
T 4
T 5
T 6
T 7
T 8
T 9
T 10
T 11
1
2
3
4
5
6

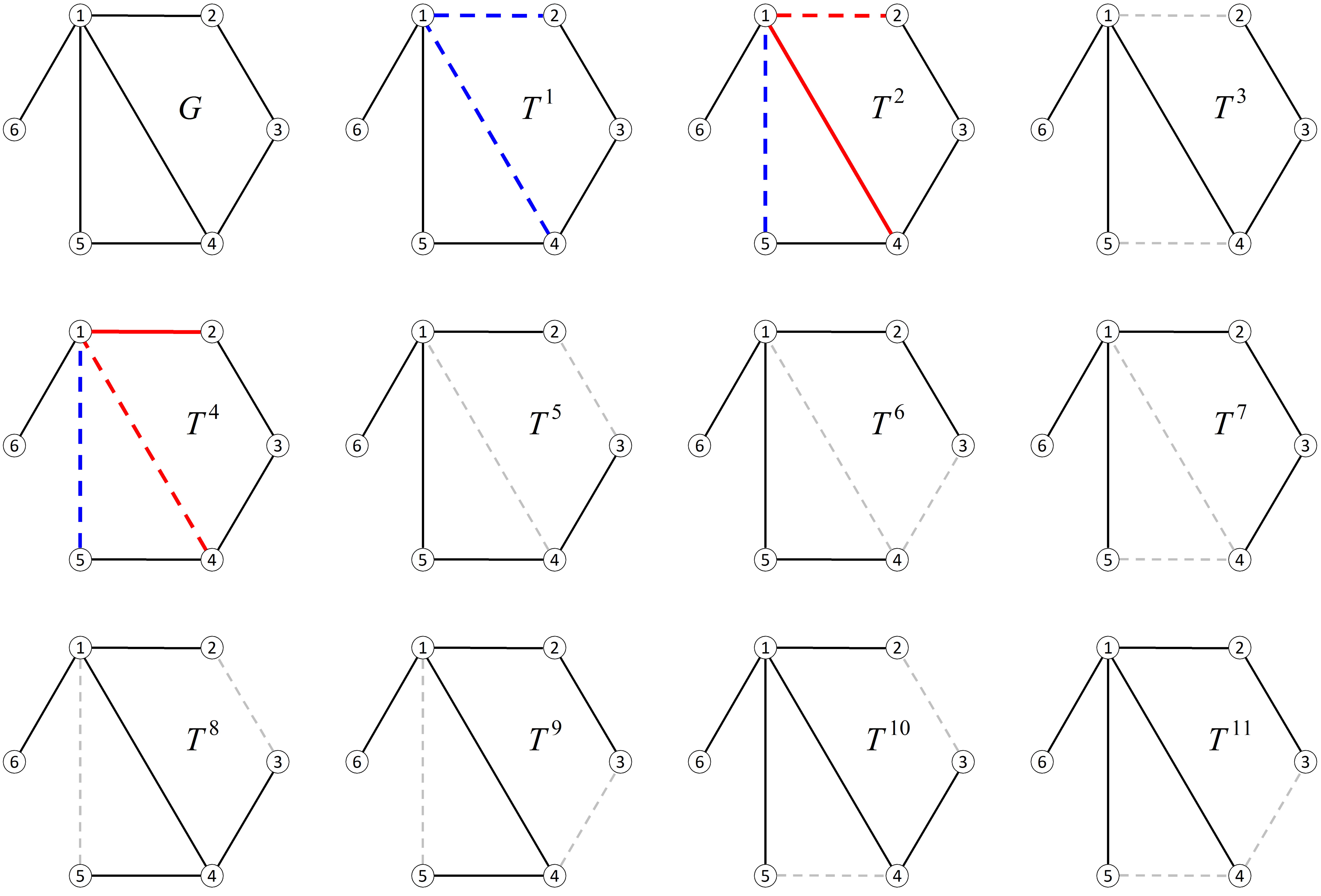

G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^10
T^11
1
2
3
4
5
6

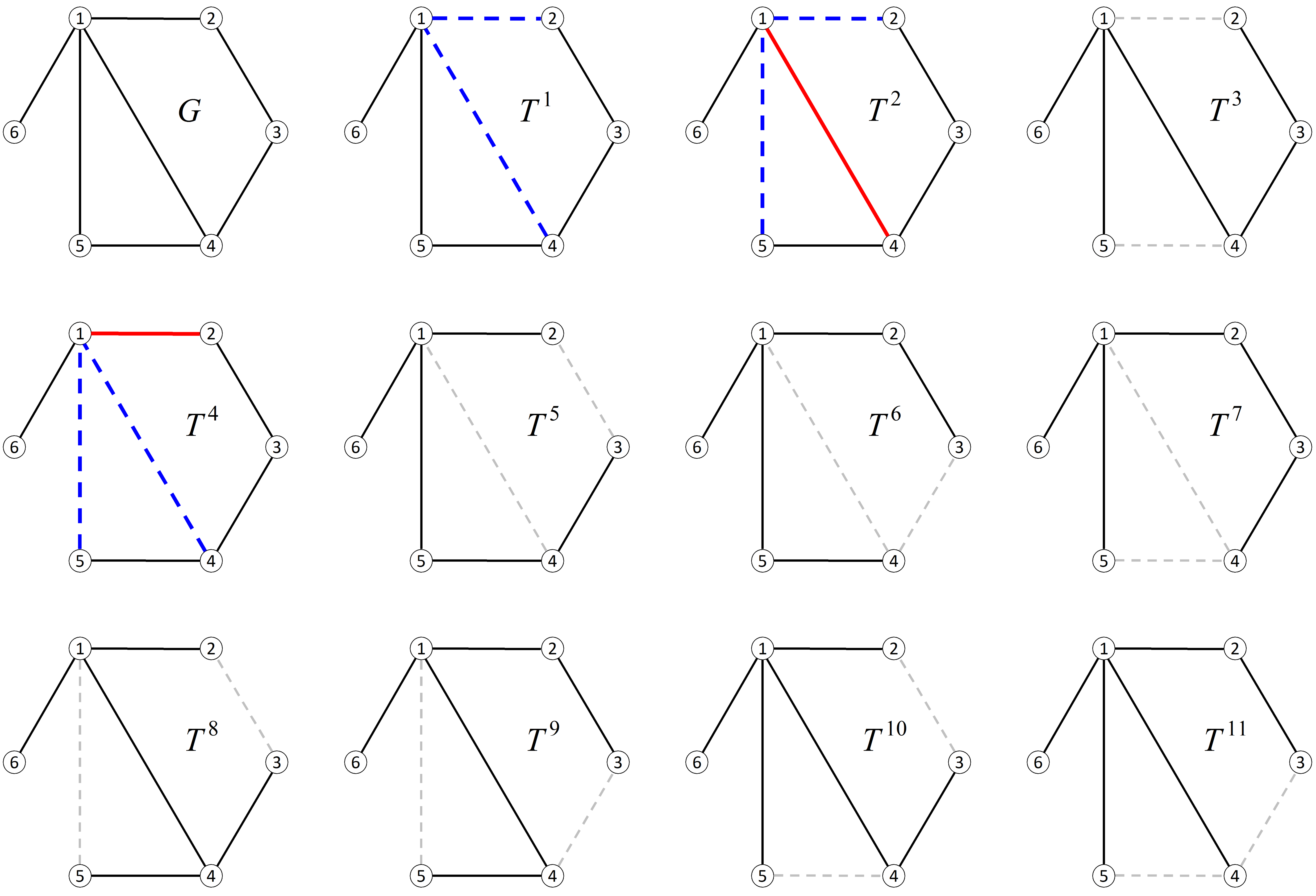


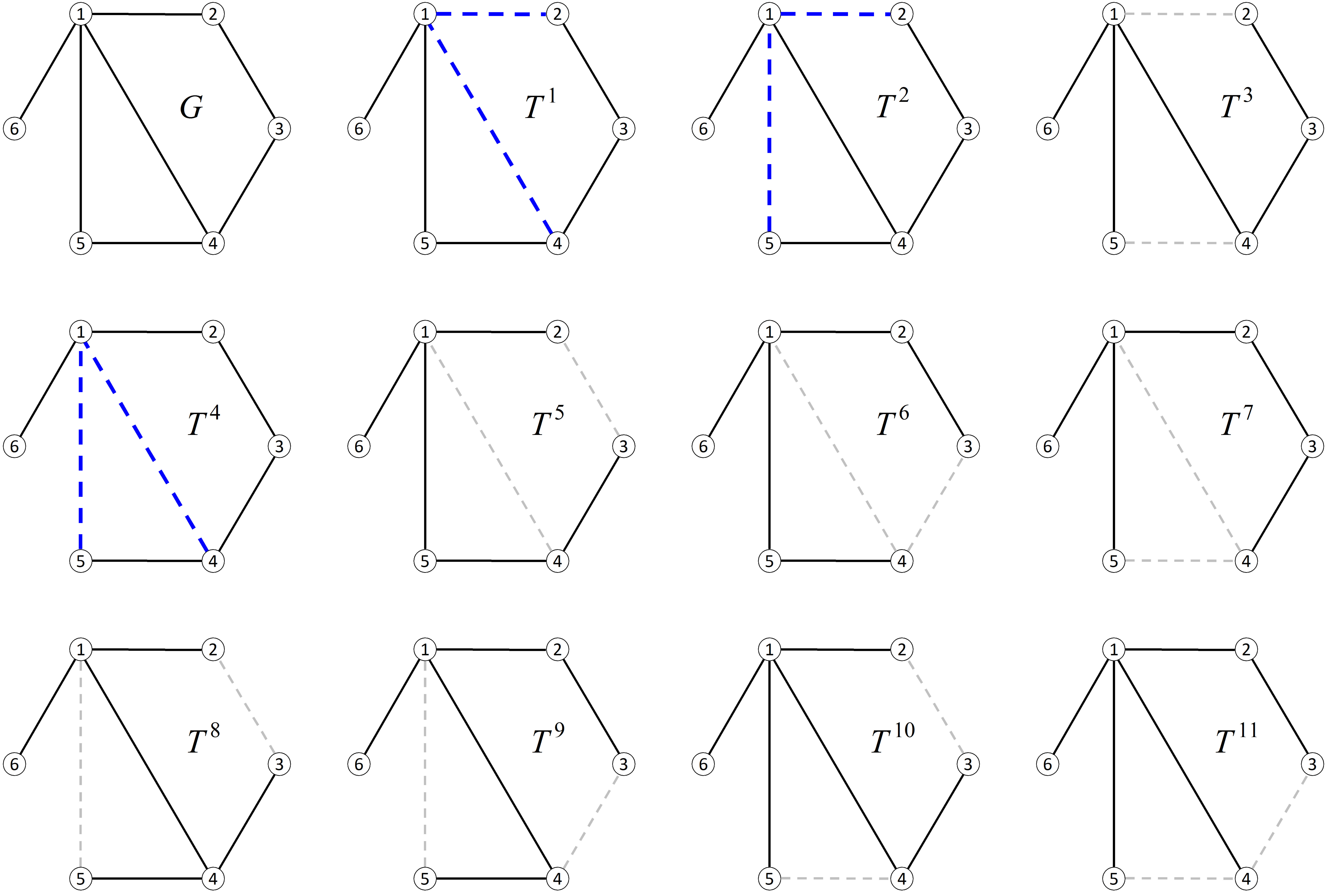
G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}



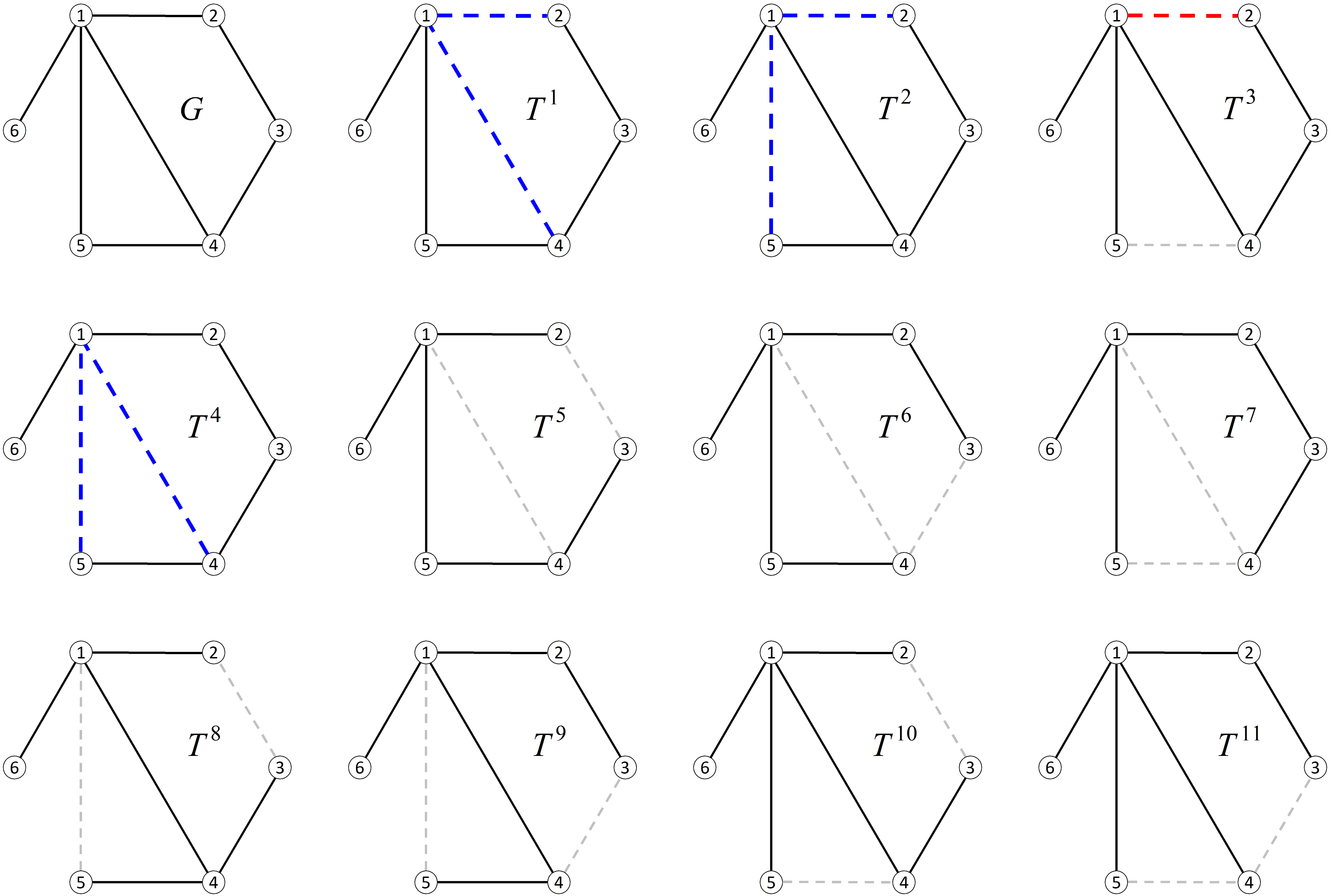
G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}

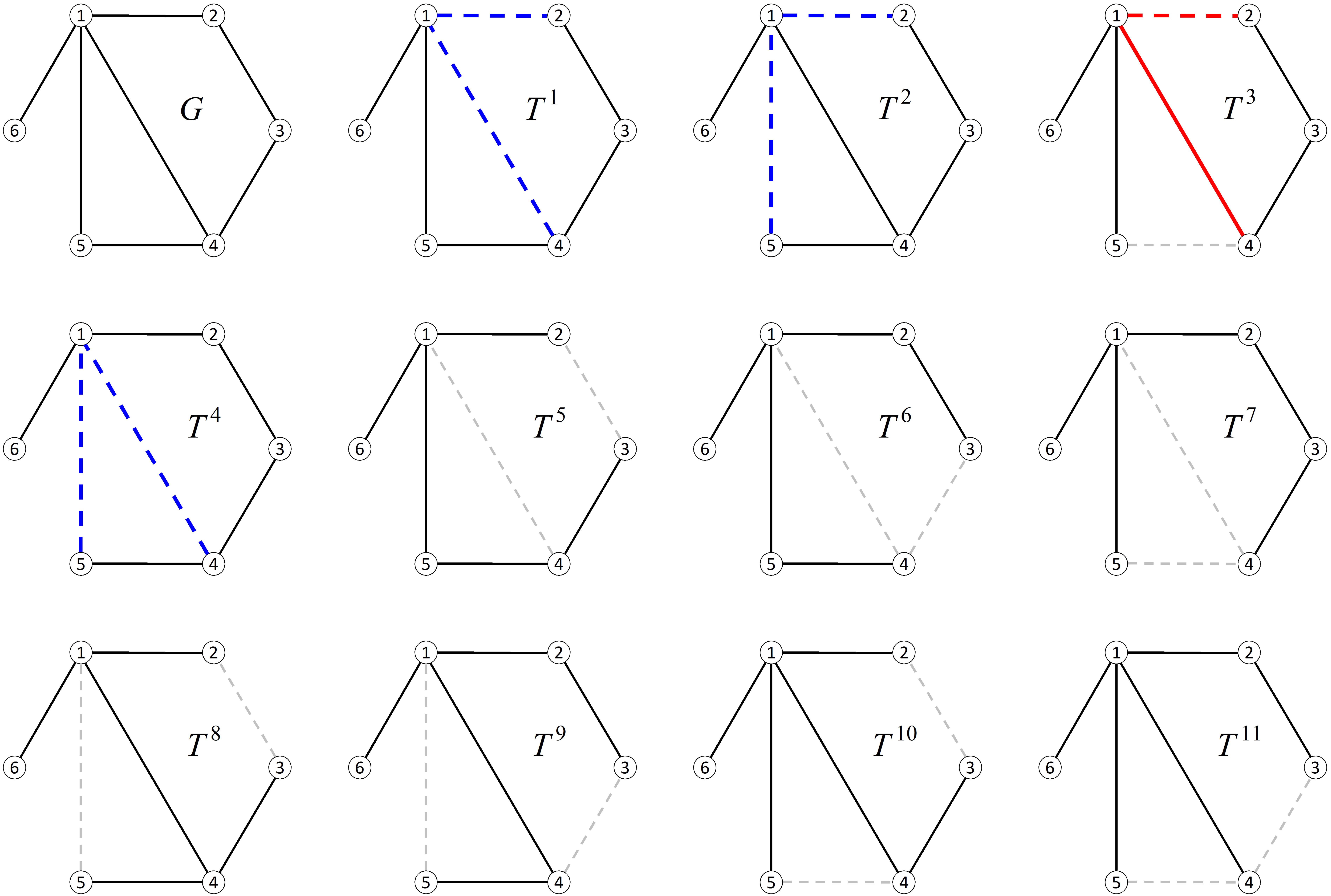

G
T1
T2
T3
T4
T5
T6
T7
T8
T9
T10
T11
1
2
3
4
5
6

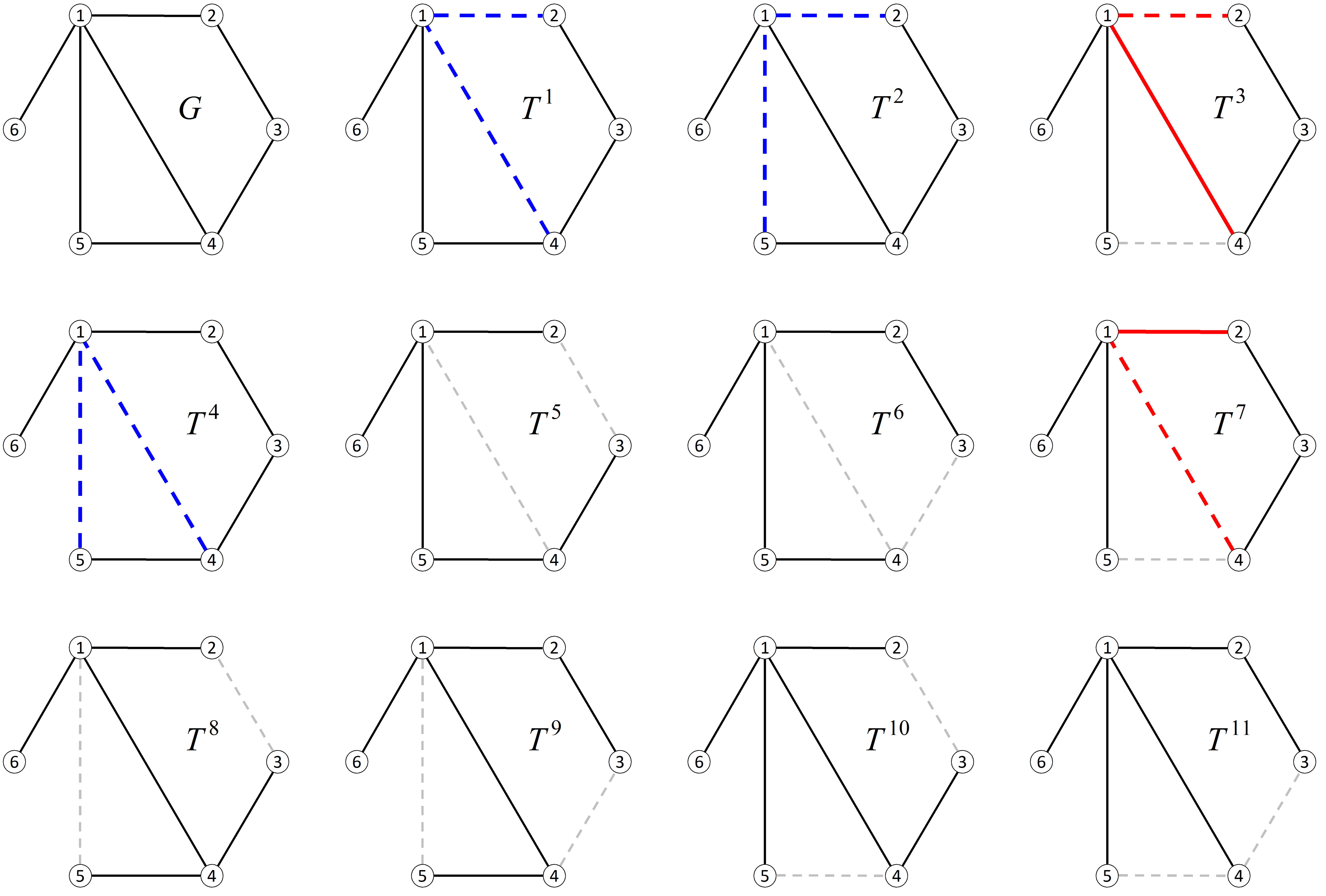

G
T 1
T 2
T 3
T 4
T 5
T 6
T 7
T 8
T 9
T 10
T 11

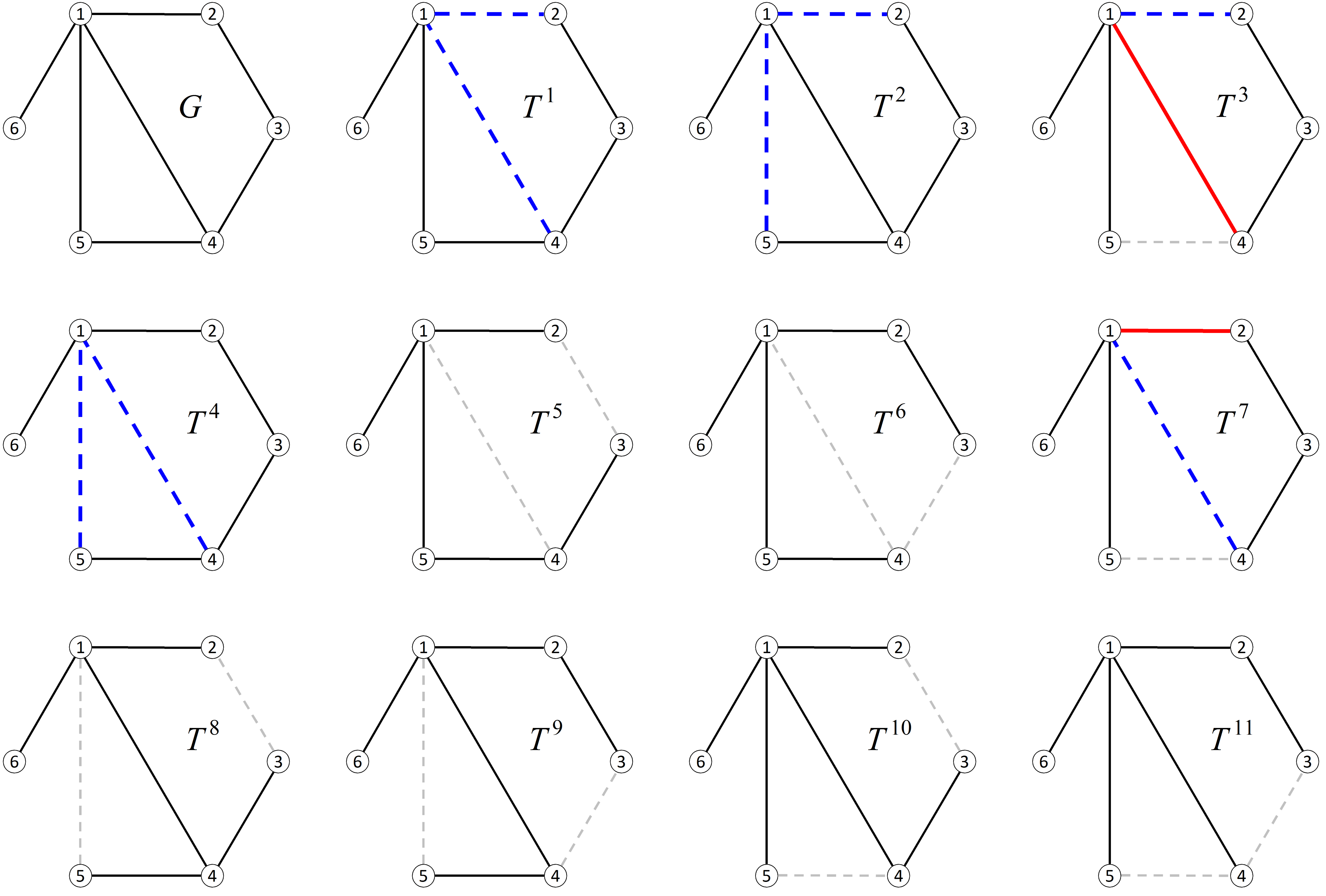


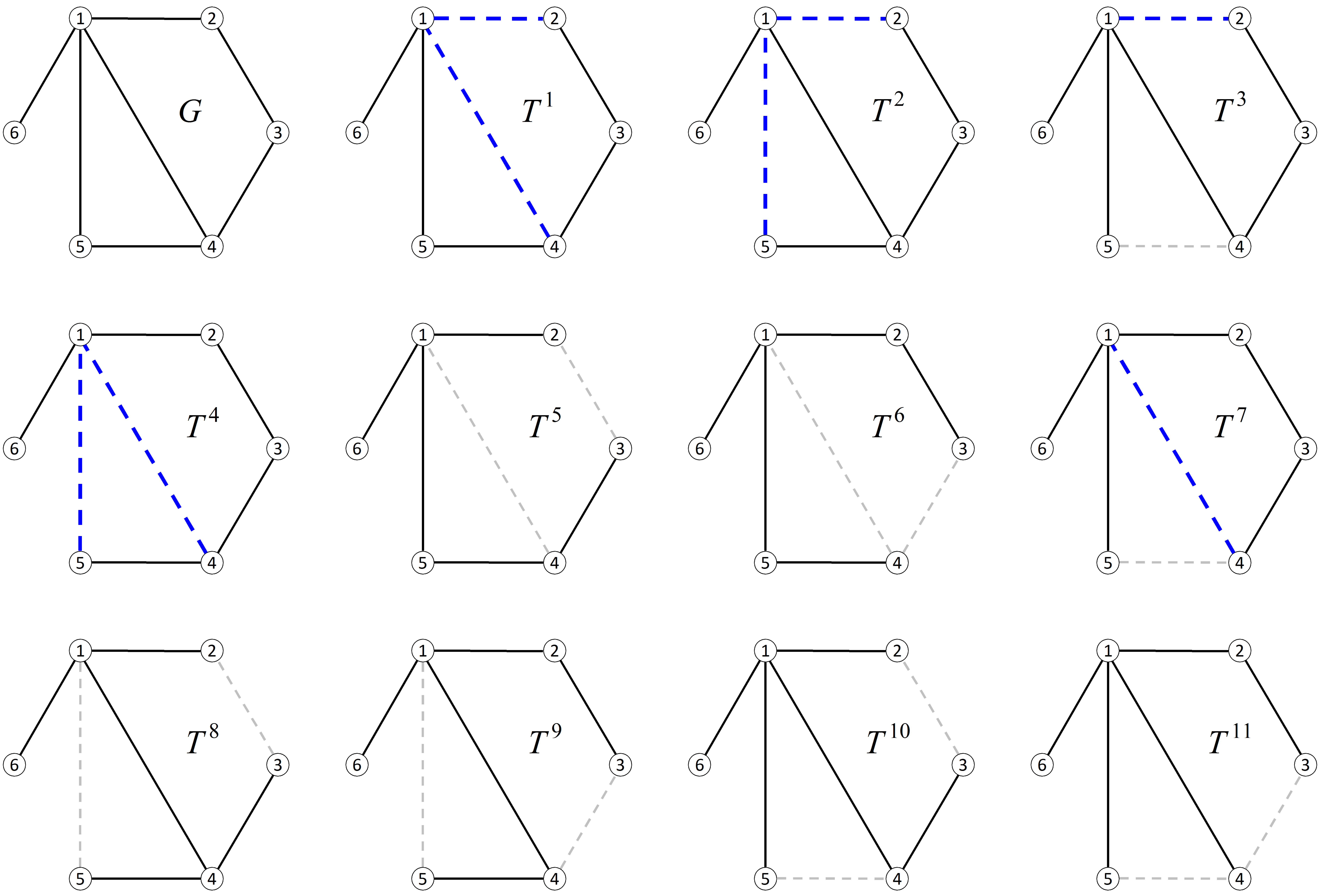
G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}

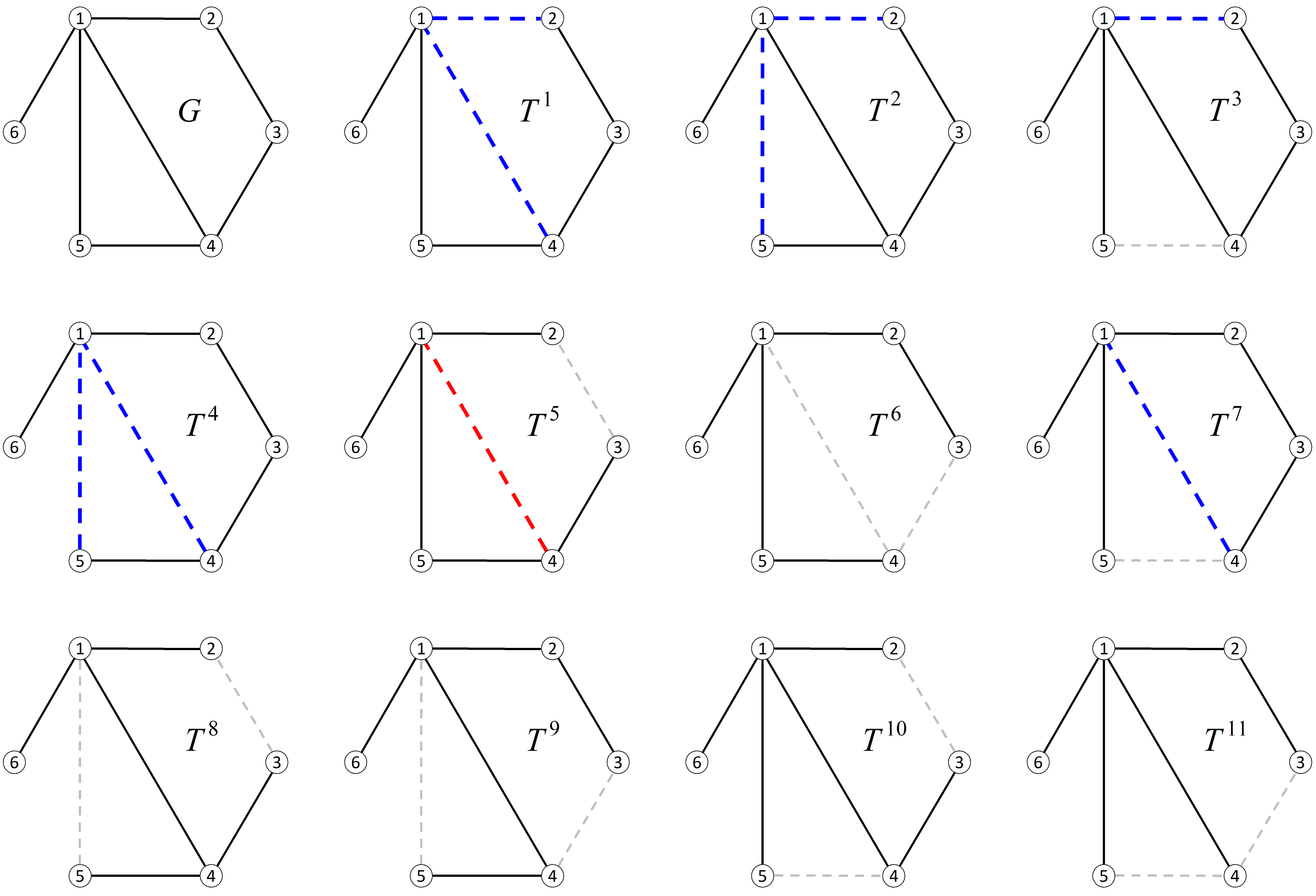

G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^10
T^11


G
T 1
T 2
T 3
T 4
T 5
T 6
T 7
T 8
T 9
T 10
T 11

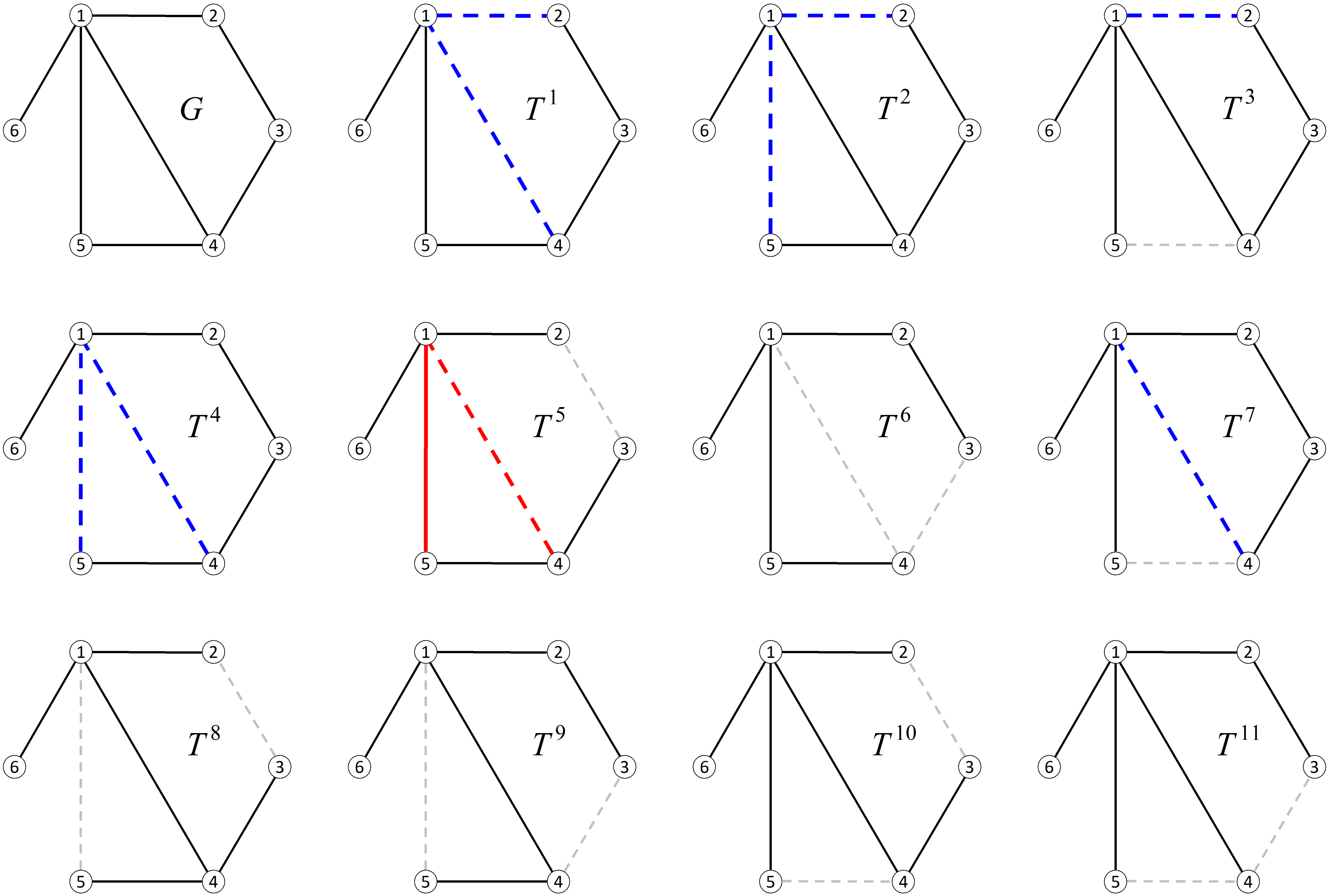

G
T 1
T 2
T 3
T 4
T 5
T 6
T 7
T 8
T 9
T 10
T 11

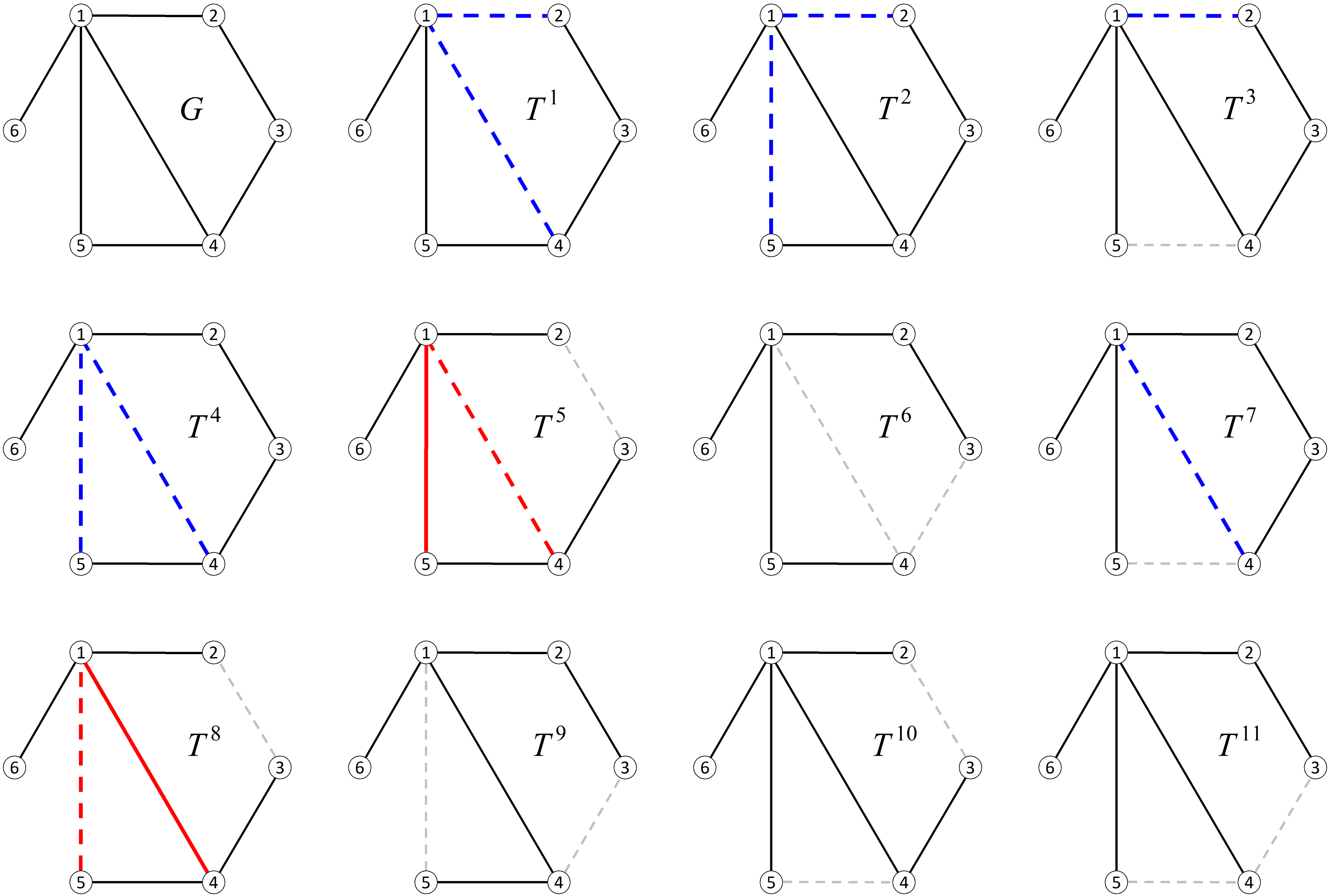

G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^10
T^11


G
T 1
T 2
T 3
T 4
T 5
T 6
T 7
T 8
T 9
T 10
T 11

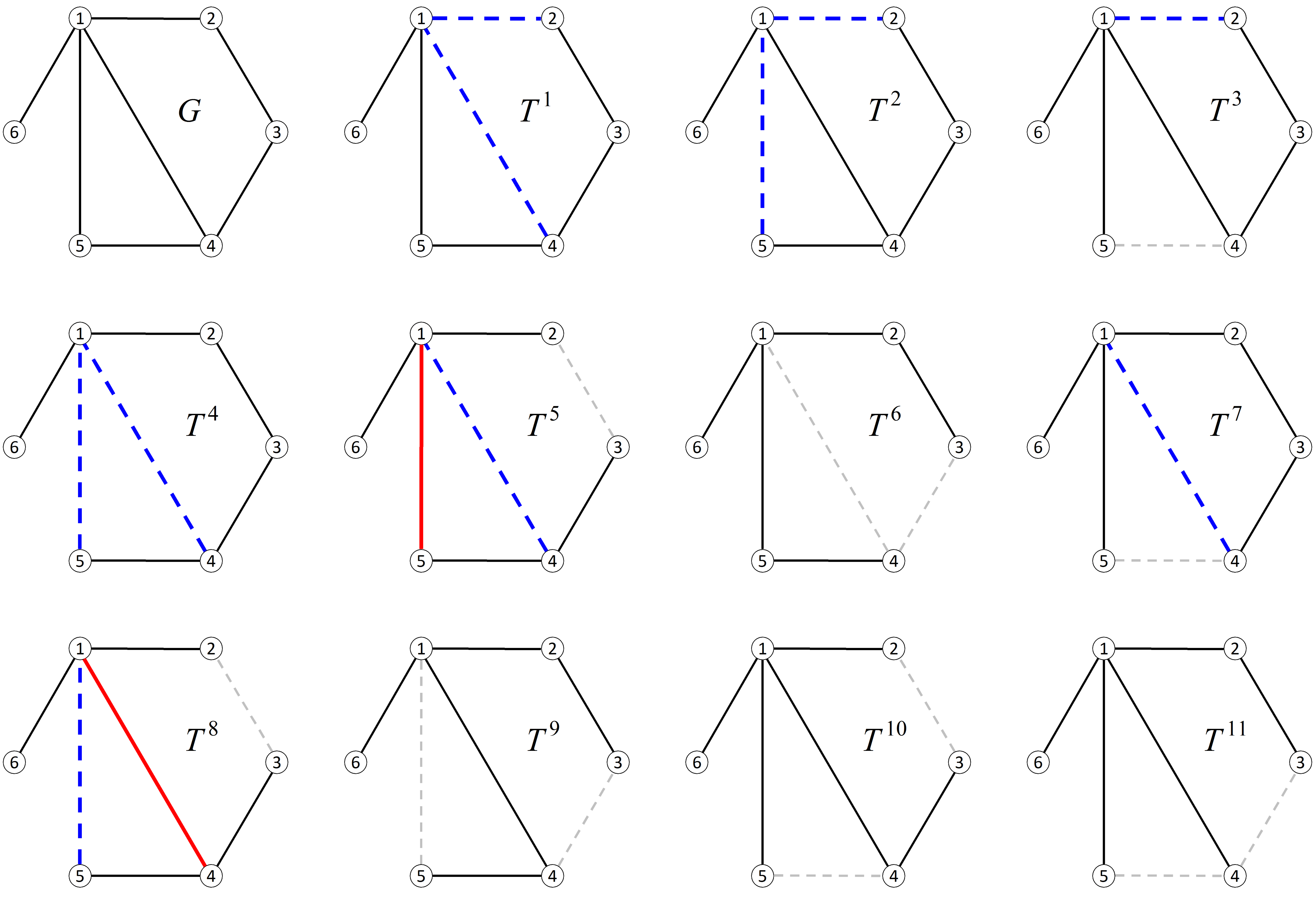


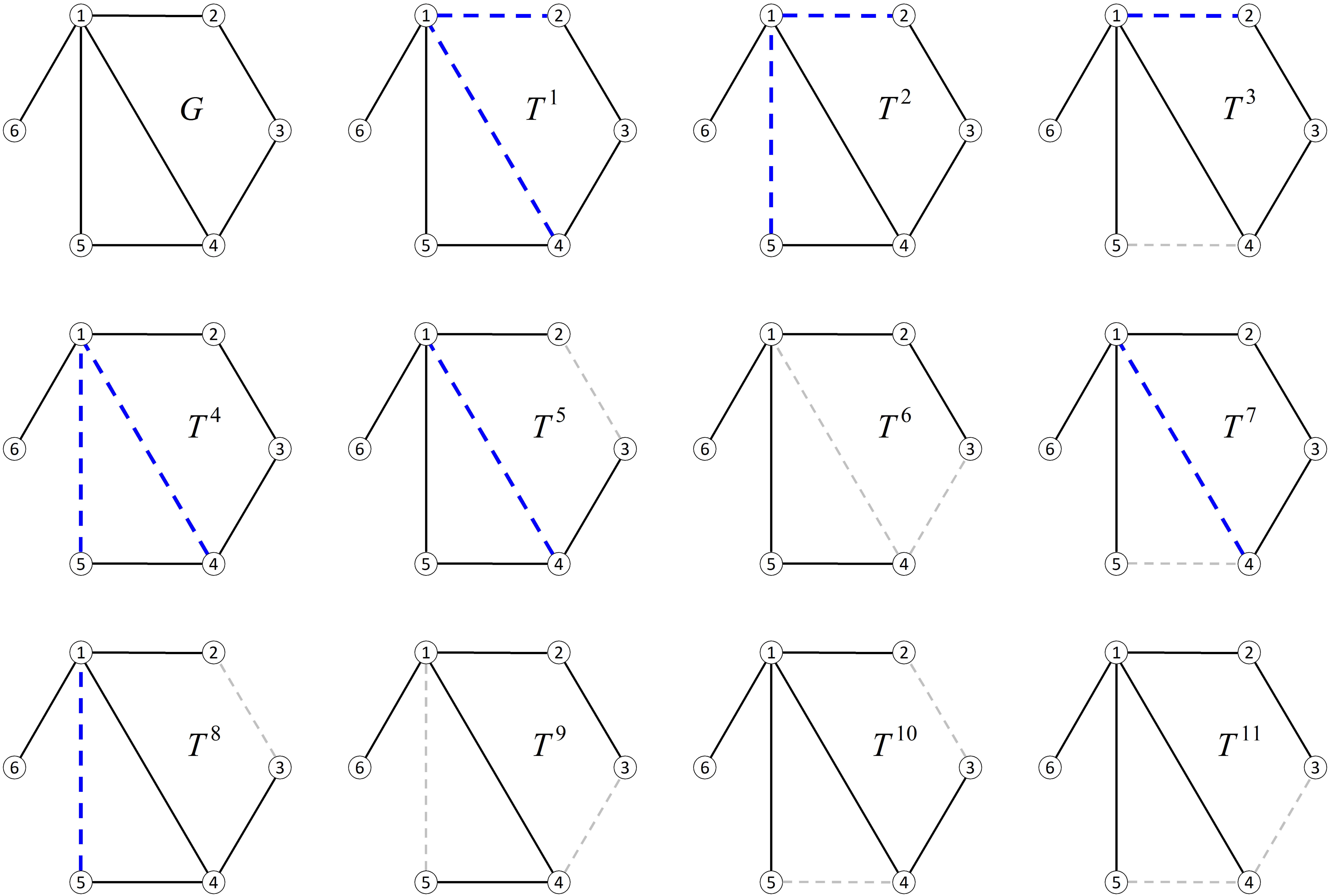
G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}

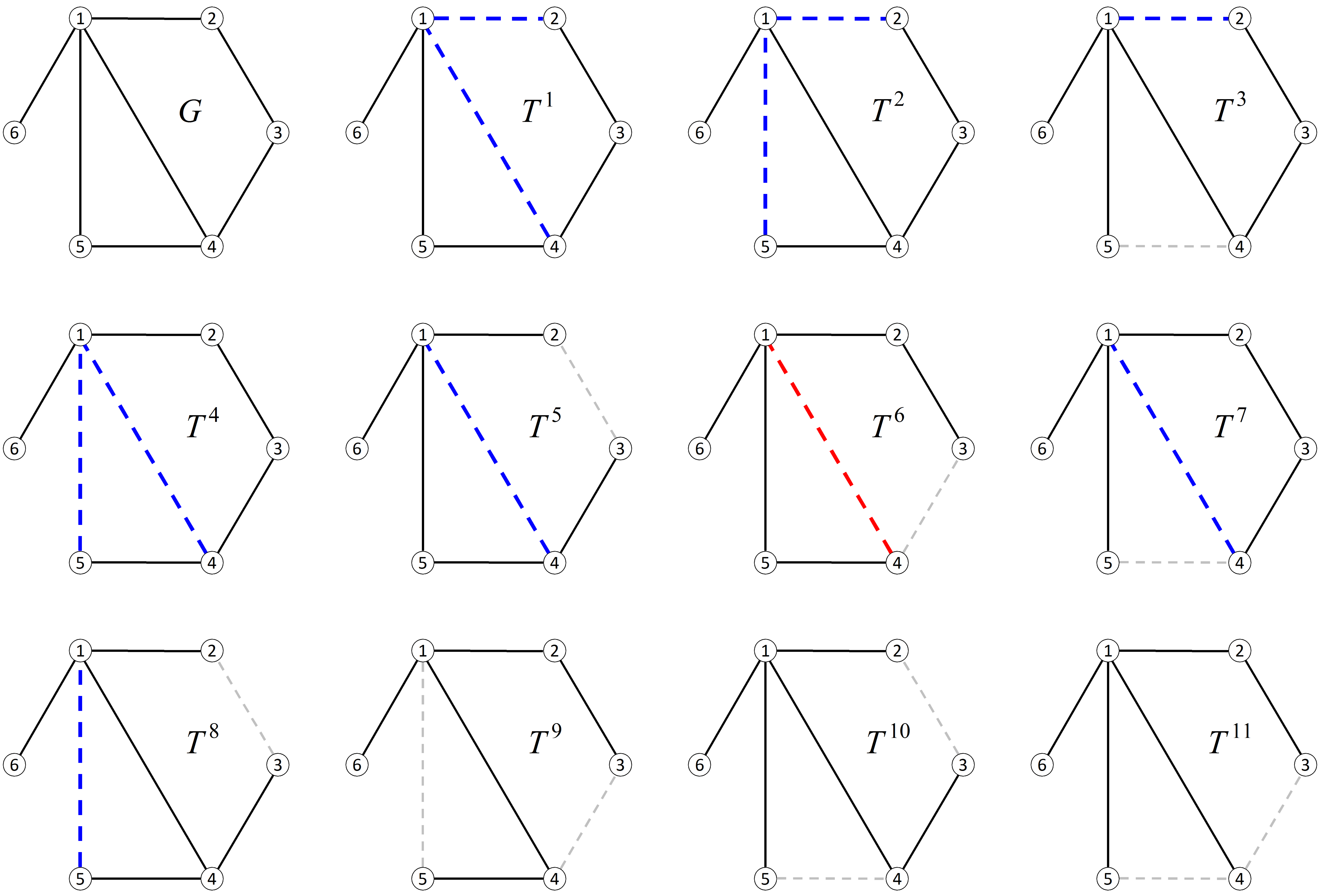

G
T1
T2
T3
T4
T5
T6
T7
T8
T9
T10
T11

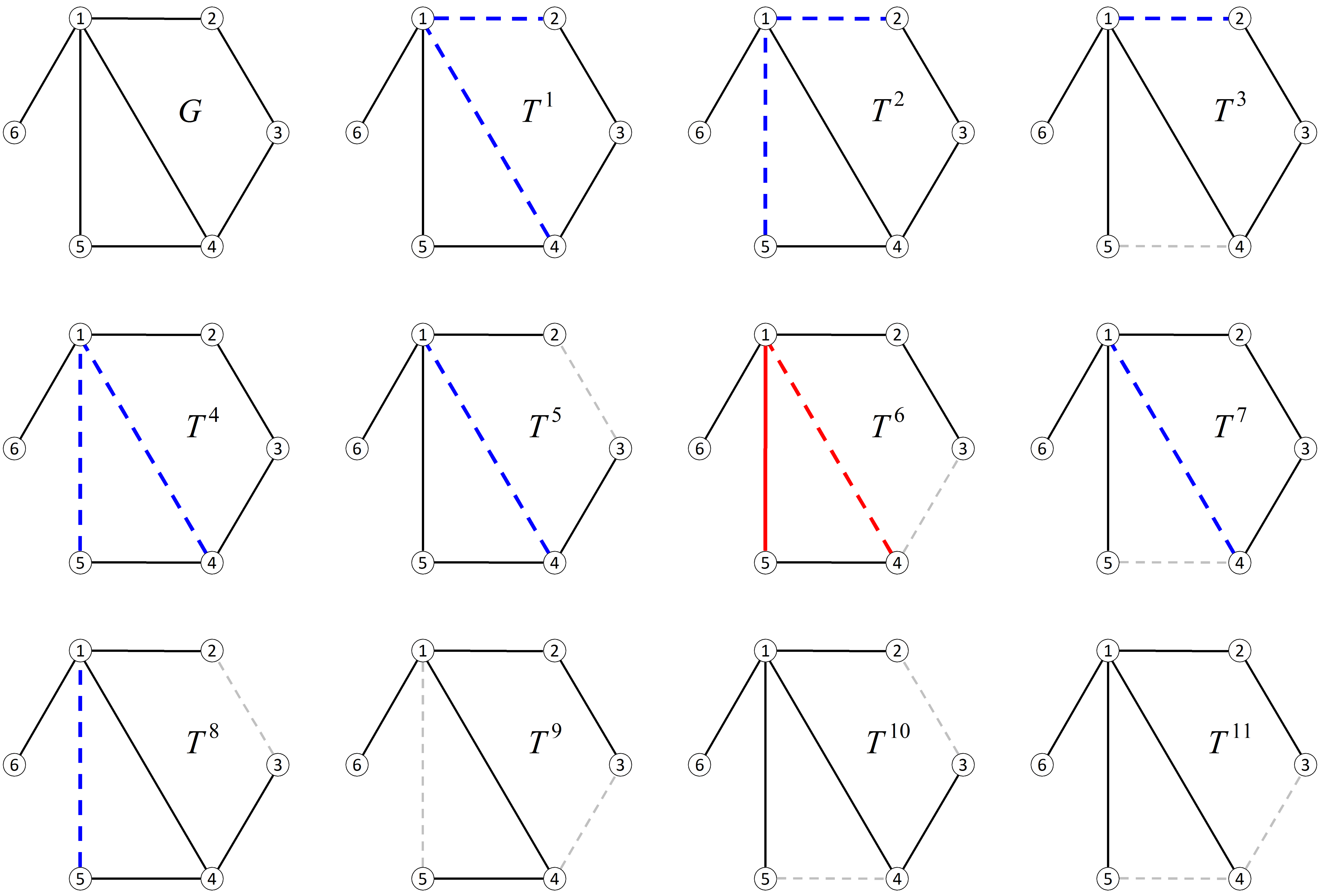


G
T1
T2
T3
T4
T5
T6
T7
T8
T9
T10
T11
1
2
3
4
5
6

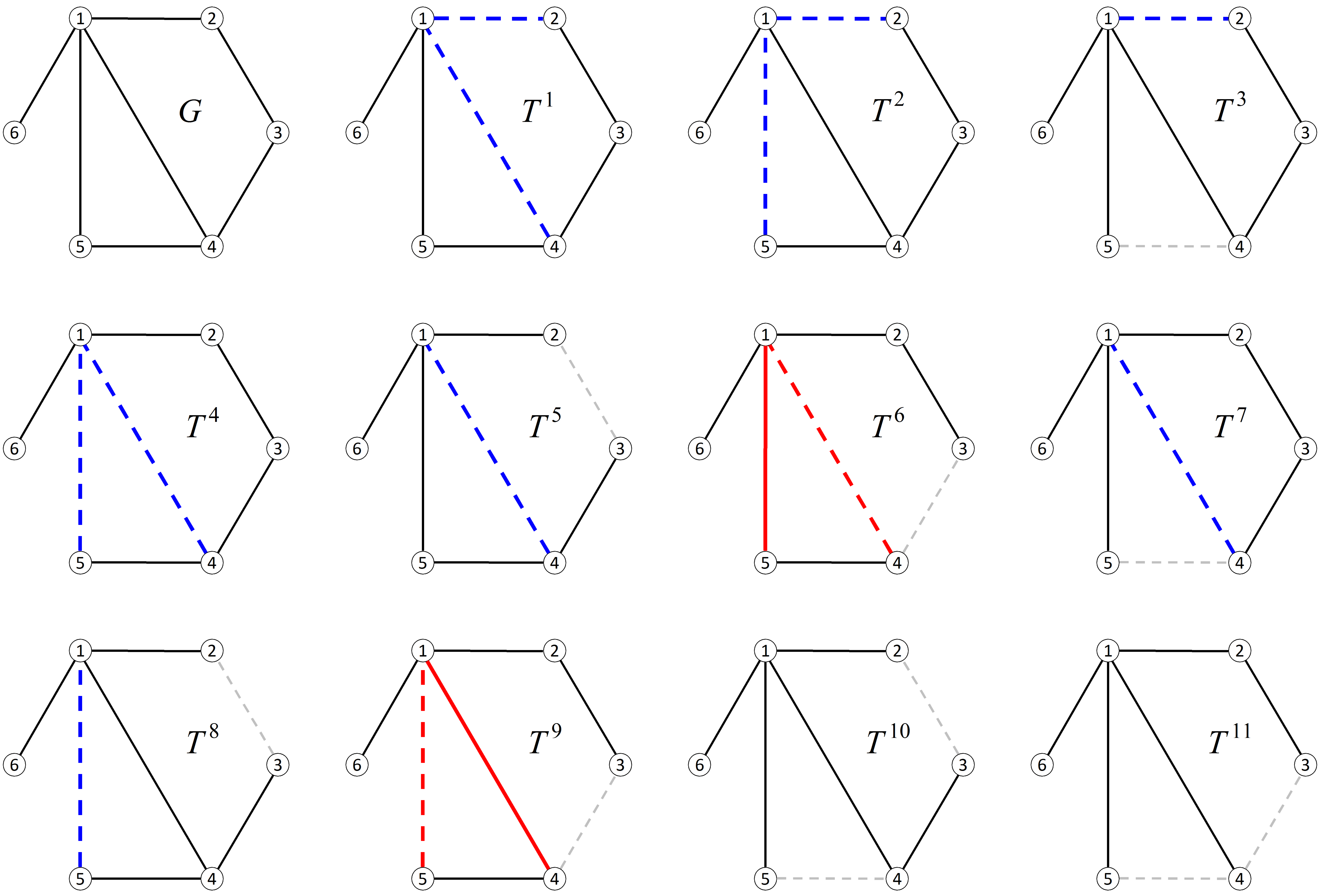

1
2
3
4
5
6
G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^10
T^11


G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}

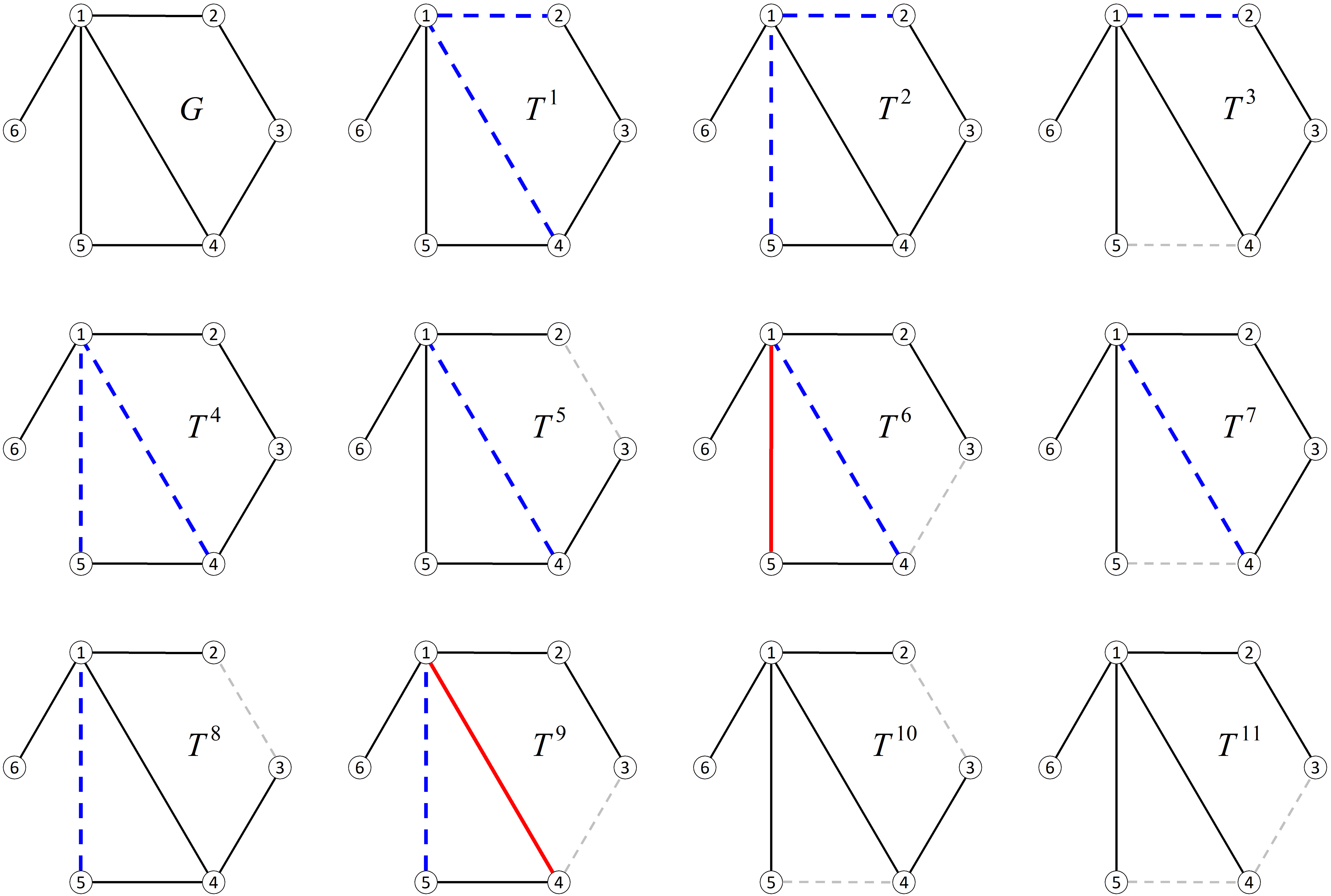

G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}
1
2
3
4
5
6

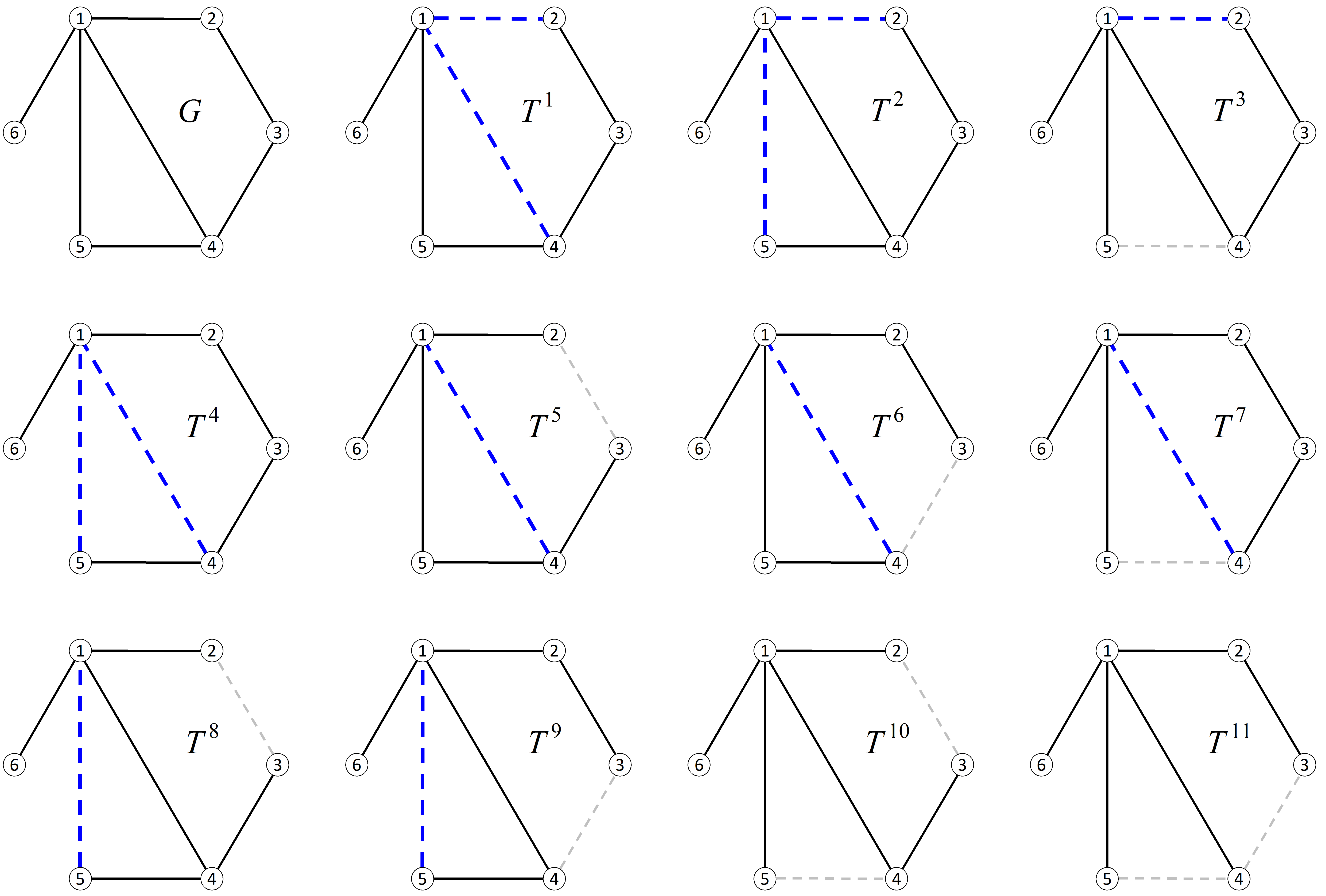


G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}



G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^10
T^11
1
2
3
4
5
6


G
T1
T2
T3
T4
T5
T6
T7
T8
T9
T10
T11


G
T^1
T^2
T^3
T^4
T^5
T^6
T^7
T^8
T^9
T^{10}
T^{11}
1
2
3
4
5
6

proof

Finally, to complete the proof, take the sum of equations

$$(\mathbf{L}\mathbf{y}^s)_i = \sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b^s_{ik} \quad \text{for all } i = 1, \ldots, n$$

for all $s = 1, 2, \ldots, S$ and apply the lemma

$$\sum_{s=1}^{S} \left(\sum_{k:e(i,k)\in E(T^s)} b_{ik} + \sum_{k:e(i,k)\in E(G)\setminus E(T^s)} b^s_{ik} \right) = S \sum_{k:e(i,k)\in E(G)} b_{ik}$$

to conclude that $\mathbf{y}^{LLS} = \frac{1}{S} \sum_{s=1}^{S} \mathbf{y}^s$. $\square$

Remarks

Complete pairwise comparison matrices ($S = n^{n-2}$) are included in our theorem as a special case, and our proof can also be considered as a second, and shorter proof of the theorem of Lundy, Siraj and Greco (2017).

Special incomplete cases, investigated by Harker (1987); van Uden (2002); Chen, Kou, Tarn, Song (2015); Bozóki (2017) are also included.

Conclusions

The equivalence of two fundamental weighting methods has been shown.

The advantages of two approaches have been united.

Main references 1/4

Tsyganok, V. (2000): Combinatorial method of pairwise comparisons with feedback. Data Recording, Storage & Processing 2:92–102 (in Ukrainian).

Tsyganok, V. (2010): Investigation of the aggregation effectiveness of expert estimates obtained by the pairwise comparison method. Mathematical and Computer Modelling, 52(3-4) 538–544

Siraj, S., Mikhailov, L., Keane, J.A. (2012): Enumerating all spanning trees for pairwise comparisons. Computers & Operations Research, 39(2) 191–199

Siraj, S., Mikhailov, L., Keane, J.A. (2012): Corrigendum to “Enumerating all spanning trees for pairwise comparisons [Comput. Oper. Res. 39(2012) 191–199]”. Computers & Operations Research, 39(9) page 2265

Main references 2/4

Lundy, M., Siraj, S., Greco, S. (2017): The mathematical equivalence of the “spanning tree” and row geometric mean preference vectors and its implications for preference analysis. European Journal of Operational Research 257(1) 197–208

Bozóki, S., Tsyganok, V. ($\geq$ 2017): The logarithmic least squares optimality of the geometric mean of weight vectors calculated from all spanning trees for (in)complete pairwise comparison matrices. Under review, https://arxiv.org/abs/1701.04265

Main references 3/4

Harker, P.T. (1987): Incomplete pairwise comparisons in the analytic hierarchy process. Mathematical Modelling 9(11)(1987), 837–848.

van Uden, E. (2002): Estimating missing data in pairwise comparison matrices. In: Bubnicki, Z., Hryniewicz, O. and Kulikowski, R. (Eds.), Operational and Systems Research in the Face to Challenge the XXI Century, Methods and Techniques in Information Analysis and Decision Making, Academic Printing House, Warsaw, pp. II-73–II-80.

Main references 4/4

Chen, K., Kou, G., Tarn, J.M., Song, J. (2015): Bridging the gap between missing and inconsistent values in eliciting preference from pairwise comparison matrices. Annals of Operations Research 235(1):155–175.

Bozóki, S., (2017): Two short proofs regarding the logarithmic least squares optimality in *Chen, K., Kou, G., Tarn, J.M., Song, J. (2015): Bridging the gap between missing and inconsistent values in eliciting preference from pairwise comparison matrices, Annals of Operations Research 235(1):155–175,* Annals of Operations Research, 253(1):707–708.

Bozóki, S., Fülöp, J. (2017): Efficient weight vectors from pairwise comparison matrices, European Journal of Operational Research (in print)
DOI 10.1016/j.ejor.2017.06.033

Thank you for attention.

bozoki.sandor@sztaki.mta.hu

http://www.sztaki.mta.hu/∼bozoki