On efficient weight vectors from pairwise comparisons

Sándor BOZÓKI[1,2], János FÜLÖP[1,3]

[1] Institute for Computer Science and Control
Hungarian Academy of Sciences (MTA SZTAKI),

[2] Corvinus University of Budapest

[3] Óbuda University, Budapest

6 August 2015

Inefficient principal right eigenvector

Example of Blanquero, Carrizosa and Conde (2006, p. 282):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 2 \\ 1/2 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/2 & 1/3 & 2 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1 \\ 2.0712416 \end{pmatrix}, \quad \mathbf{w}^{*} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1.003 \\ 2.0712416 \end{pmatrix}.$$

i	a_{i3}	x_{i3}^{EM}	x_{i3}^{*}	$\lvert a_{i3} - x_{i3}^{EM} \rvert$	$\lvert a_{i3} - x_{i3}^{*} \rvert$
1	6	6.01438057	5.99639139	0.01438057	0.00360859
2	4	4.26049429	4.24775103	0.26049429	0.24775103
3	1	1	1	0	0
4	2	2.07124160	2.06504646	0.07124160	0.06504646

Definitions and notations

$\mathcal{PCM}_n$ denotes the set of pairwise comparison matrices of size $n \times n$.

$\lambda_{\max}(\mathbf{A})$ denotes the dominant eigenvalue of pairwise comparison matrix $\mathbf{A}$ of size $n \times n$.

$\mathbf{w}^{EM(\mathbf{A})}$, also called EM *weight vector*, denotes the principal right eigenvector of $\mathbf{A}$ corresponding to $\lambda_{\max}(\mathbf{A})$.

$\mathbf{w}^{EM(\mathbf{A})}$ is usually normalized to 1, that is, $\sum\limits_{i=1}^{n} w_i^{EM(\mathbf{A})} = 1.$

$\mathbf{X}^{EM(\mathbf{A})} = \mathbf{X}^{EM} \stackrel{\text{def}}{=} \left[\frac{w_i^{EM(\mathbf{A})}}{w_j^{EM(\mathbf{A})}} \right]_{i,j=1,\ldots,n}$ is the consistent pairwise comparison matrix generated by $\mathbf{w}^{EM(\mathbf{A})}$.
It is the approximation of $\mathbf{A}$ by the eigenvector method.

The multi-objective optimization problem is as follows:

$$\min_{x_i > 0 \ \forall i} \left(\left| a_{ij} - \frac{x_i}{x_j} \right| \right)_{i \neq j}$$

Efficiency

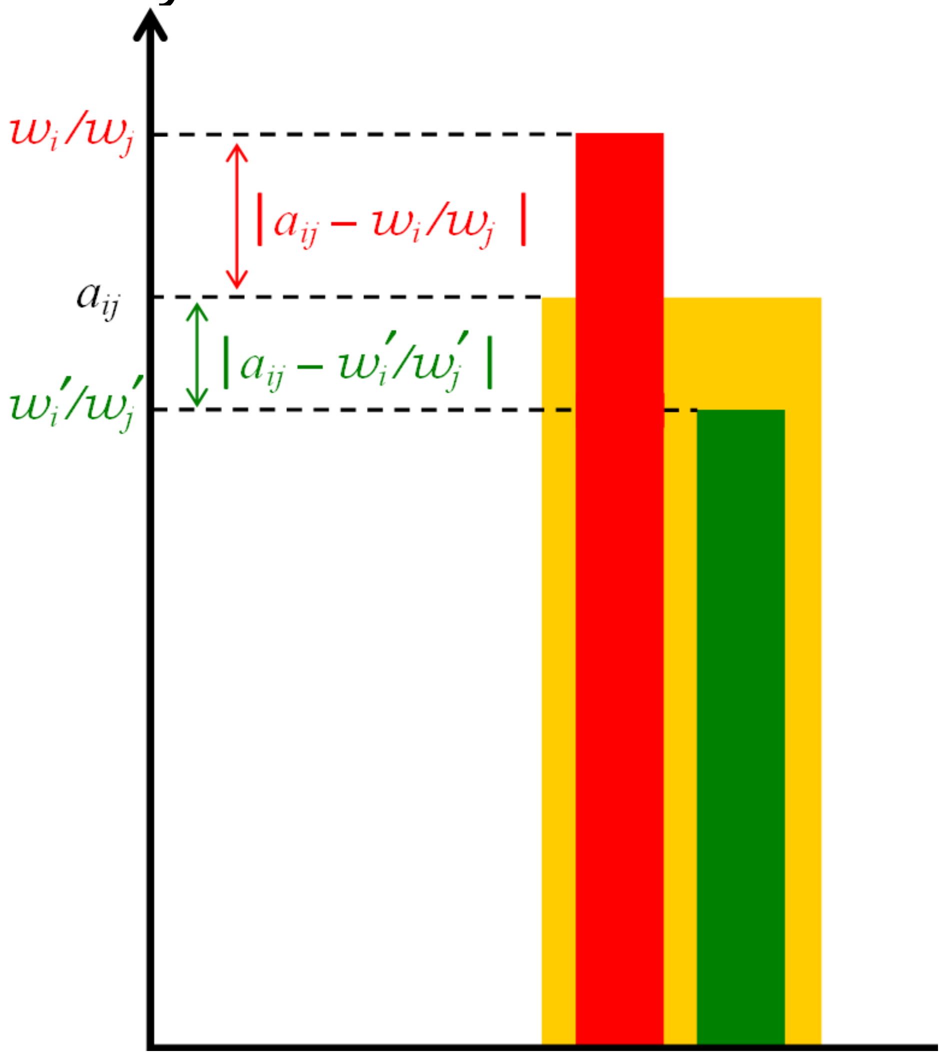
Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$ be an $n \times n$ pairwise comparison matrix and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector.

Definition: weight vector $\mathbf{w}$ is called *efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n,$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \leq k, \ell \leq n.$$

An efficient weight vector cannot be improved such that every element of the pairwise comparison matrix is approximated at least as good, and at least one element is approximated strictly better.



Local efficiency

Definition: weight vector $\mathbf{w}$ is called *locally efficient,* if there is a neighborhood of $\mathbf{w}$, denoted by $V(\mathbf{w})$, such that there exists no positive weight vector $\mathbf{w}' \in V(\mathbf{w})$ fulfilling

$$\left| a_{ij} - \frac{w_i'}{w_j'} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i, j \leq n,$$

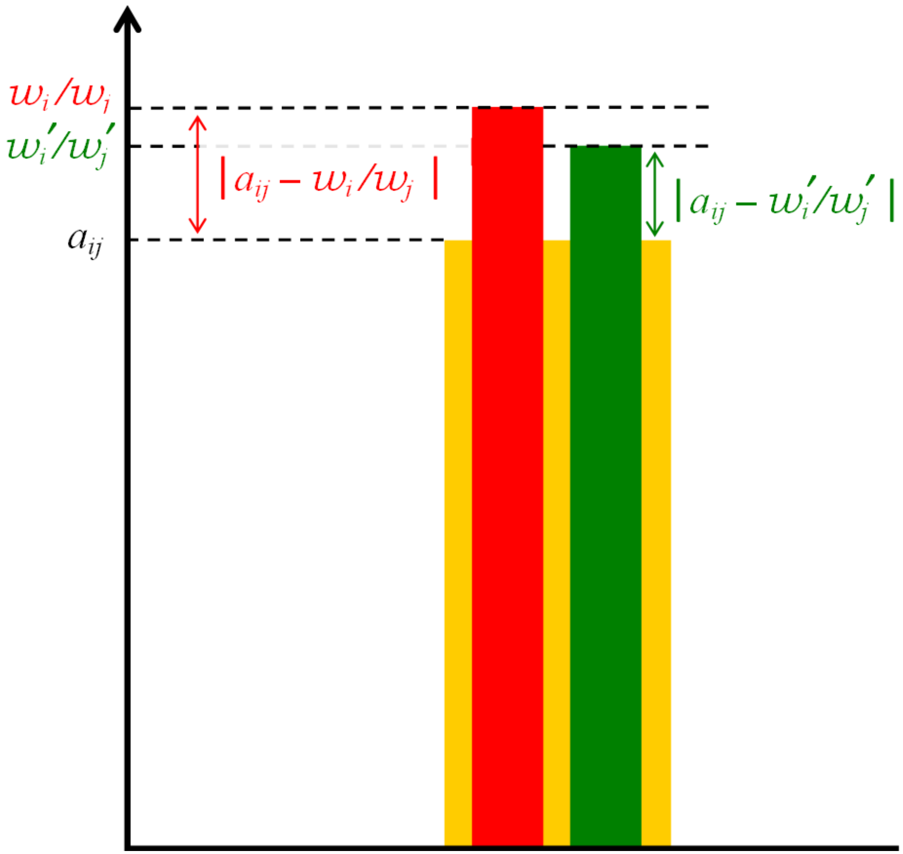
$$\left| a_{k\ell} - \frac{w_k'}{w_\ell'} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \leq k, \ell \leq n.$$

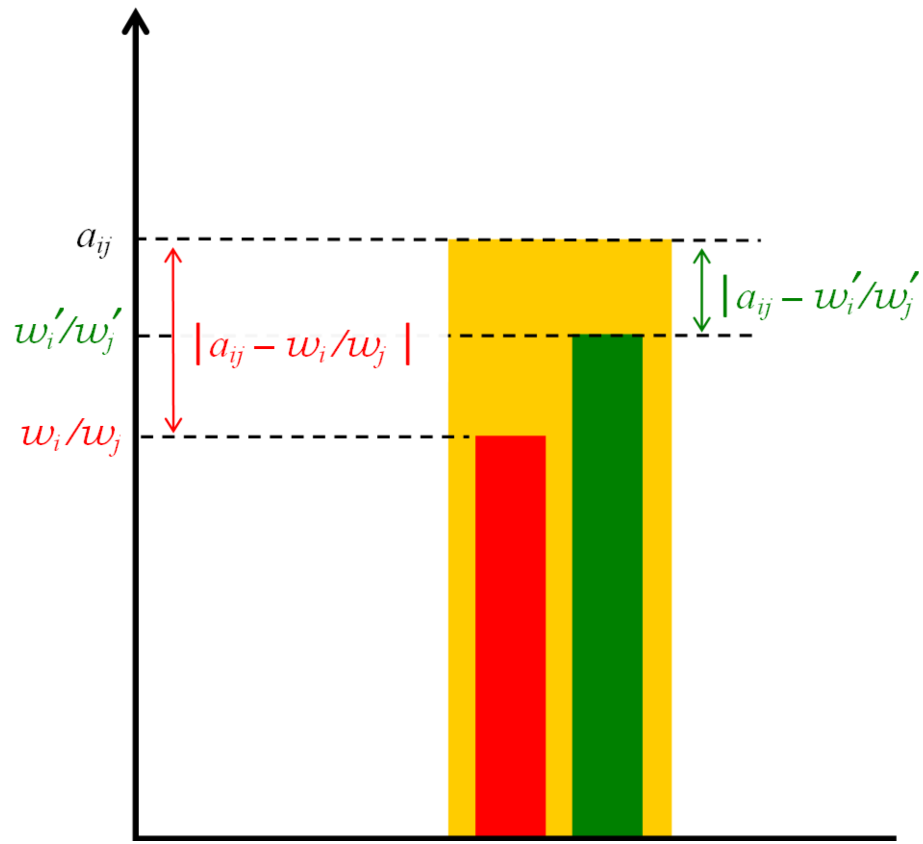
Internal efficiency

Definition: weight vector $\mathbf{w}$ is called *internally efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ such that

$$\left.\begin{array}{lcl} a_{ij} \leq \frac{w_i}{w_j} & \Longrightarrow & a_{ij} \leq \frac{w'_i}{w'_j} \leq \frac{w_i}{w_j} \\ a_{ij} \geq \frac{w_i}{w_j} & \Longrightarrow & a_{ij} \geq \frac{w'_i}{w'_j} \geq \frac{w_i}{w_j} \end{array}\right\} \text{ for all } 1 \leq i, j \leq n, \text{ and}$$

$$\left.\begin{array}{lcl} a_{k\ell} \leq \frac{w_k}{w_\ell} & \Longrightarrow & \frac{w'_k}{w'_\ell} < \frac{w_k}{w_\ell} \\ a_{k\ell} \geq \frac{w_k}{w_\ell} & \Longrightarrow & \frac{w'_k}{w'_\ell} > \frac{w_k}{w_\ell} \end{array}\right\} \text{ for some } 1 \leq k, \ell \leq n.$$


w_i/w_j
w_i'/w_j'
a_{ij}
$|a_{ij} - w_i/w_j|$
$|a_{ij} - w_i'/w_j'|$


a_{ij}
w_i'/w_j'
w_i/w_j
$|a_{ij} - w_i/w_j|$
$|a_{ij} - w_i'/w_j'|$

efficient = locally efficient = internally efficient

Proposition (Blanquero, Carrizosa and Conde, 2006; Bozóki, Fülöp, 2015):

Definitions of

- efficiency
- local efficiency
- internal efficiency

are equivalent.

Following the way of Blanquero, Carrizosa and Conde (2006)

$$\min_{x_i > 0\ \forall i} \left(\left| a_{ij} - \frac{x_i}{x_j} \right| \right)_{i \neq j}$$

Denote $\varepsilon_{ij} := \left| \frac{w_i}{w_j} - a_{ij} \right|$.

Proposition: $\mathbf{w}$ is efficient if and only if for any pair of indices $k, \ell = 1, 2, \ldots, n,\ k \neq \ell$, $\mathbf{w}$ is an optimal solution to the fractional optimization problem

$$\begin{aligned} &\inf \left| \frac{x_k}{x_\ell} - a_{k\ell} \right| \\ &\quad \left| \frac{x_i}{x_j} - a_{ij} \right| \leq \varepsilon_{ij} \qquad \text{for all pairs } (i,j) \neq (k,\ell) \\ &x_1, x_2, \ldots, x_n > 0. \end{aligned}$$

$$\left|\frac{x_i}{x_j} - a_{ij}\right| \leq \varepsilon_{ij} \Longleftrightarrow |x_i - a_{ij}x_j| \leq \varepsilon_{ij}x_j \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} x_i - a_{ij}x_j + t_{ij} & \leq & \varepsilon_{ij}x_j \\ a_{ij}x_j - x_i + t_{ij} & \leq & \varepsilon_{ij}x_j \\ t_{ij} & \geq & 0 \end{cases}$$

$$\varepsilon_{ij} := \left| \frac{w_i}{w_j} - a_{ij} \right|$$

Let multipliers $\beta_{ij} > 0 \ (i, j = 1, 2, \ldots, n)$ be arbitrary.

$$\begin{aligned}
\max \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} t_{ij} & \\
x_i - (a_{ij} + \varepsilon_{ij}) x_j + t_{ij} \le 0, & \qquad i, j = 1, 2, \ldots, n \\
-x_i + (a_{ij} - \varepsilon_{ij}) x_j + t_{ij} \le 0, & \qquad i, j = 1, 2, \ldots, n \\
x_1 = 1, & \qquad \qquad \qquad \qquad \text{(LP)} \\
x_i \ge 0, & \qquad i = 2, 3, \ldots, n \\
t_{ij} \ge 0, & \qquad i, j = 1, 2, \ldots, n \\
t_{ii} = 0, & \qquad i = 1, 2, \ldots, n
\end{aligned}$$

Variables: $x_i, \ i = 1, 2, \ldots, n,$ and $t_{ij}, \ i, j = 1, 2, \ldots, n$.

$$\varepsilon_{ij} := \left| \frac{w_i}{w_j} - a_{ij} \right|$$

$$\begin{aligned}
\max \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} t_{ij} & \\
x_i - (a_{ij} + \varepsilon_{ij}) x_j + t_{ij} \le 0, & \qquad i, j = 1, 2, \ldots, n \\
-x_i + (a_{ij} - \varepsilon_{ij}) x_j + t_{ij} \le 0, & \qquad i, j = 1, 2, \ldots, n \\
x_1 = 1, & \\
x_i \ge 0, & \qquad i = 2, 3, \ldots, n \\
t_{ij} \ge 0, & \qquad i, j = 1, 2, \ldots, n \\
t_{ii} = 0, & \qquad i = 1, 2, \ldots, n
\end{aligned} \tag{LP}$$

Theorem: (Bozóki, Fülöp, 2015): The optimum value of (LP) is finite and nonnegative. Furthermore, $\mathbf{w}$ is efficient if and only if the optimum value of (LP) is zero.

Note that an optimal solution to (LP) does not necessarily yields an efficient weight vector.

However, we have formulated a more sophisticated LP that finds an efficient weight vector (that improves the starting weight vector) in at most n steps.

Characterization of efficiency

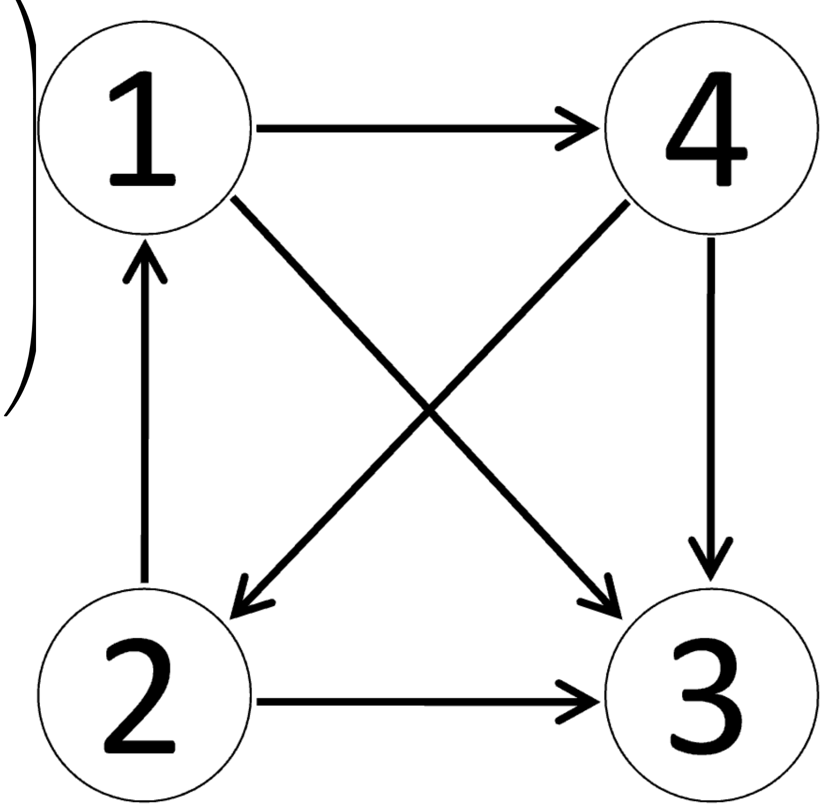
Definition: Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathcal{PCM}_n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector. Directed graph $(V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is defined as follows: $V = \{1, 2, \ldots, n\}$ and

$$\overrightarrow{E} = \left\{ \text{arc}(i \rightarrow j) \,\middle|\, \frac{w_i}{w_j} \geq a_{ij}, i \neq j \right\}.$$

Theorem (Blanquero, Carrizosa and Conde, 2006): Weight vector $\mathbf{w}$ is efficient if and only if $(V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is strongly connected, that is, there exist directed paths from i to j and from j to i for all pairs of $i \neq j$ nodes.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 2 \\ 1/2 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/2 & 1/3 & 2 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1 \\ 2.0712416 \end{pmatrix}$$

$$\mathbf{X}^{EM} = \begin{pmatrix} 1 & {\color{blue}1.41} & {\color{red}6.01} & {\color{red}2.90} \\ {\color{red}0.71} & 1 & {\color{red}4.26} & {\color{blue}2.06} \\ {\color{blue}0.1663} & {\color{blue}0.23} & 1 & {\color{blue}0.48} \\ {\color{blue}0.34} & {\color{red}0.49} & {\color{red}2.07} & 1 \end{pmatrix}$$



Special cases

Efficient principal right eigenvector:

- simple perturbed PCM
- double perturbed PCM

Inefficient principal right eigenvector:

- PCM with arbitrarily small inconsistency
- Numerical examples

Simple perturbed PCM

Consider a consistent matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & x_1 & x_2 & \dots & x_{n-1} \\ \frac{1}{x_1} & 1 & \frac{x_2}{x_1} & \dots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \dots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \dots & 1 \end{pmatrix} \in \mathcal{PCM}_n,$$

then perturb a single element and its reciprocal. The perturbation is realized by a multiplication by $\delta > 0, \delta \neq 1$, while the reciprocal element is divided by δ.

Simple perturbed PCM: $\mathbf{w}^{EM}$ is efficient

$$\mathbf{A}_\delta = \begin{pmatrix} 1 & \delta x_1 & x_2 & \dots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \dots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \dots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \dots & 1 \end{pmatrix} \in \mathcal{PCM}_n.$$

Theorem (Ábele-Nagy, Bozóki, 2015):
The principal right eigenvector of a simple perturbed pairwise comparison matrix is efficient.

Proof is based on the explicit formulas of $\mathbf{w}^{EM}$.

Double perturbed PCM ($n \geq 4$)

$$\begin{pmatrix} 1 & \delta x_1 & \gamma x_2 & x_3 & \dots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \frac{x_3}{x_1} & \dots & \frac{x_{n-1}}{x_1} \\ \frac{1}{\gamma x_2} & \frac{x_1}{x_2} & 1 & \frac{x_3}{x_2} & \dots & \frac{x_{n-1}}{x_2} \\ \frac{1}{x_3} & \frac{x_1}{x_3} & \frac{x_2}{x_3} & 1 & \dots & \frac{x_{n-1}}{x_3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \frac{x_3}{x_{n-1}} & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \delta x_1 & x_2 & x_3 & \dots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \frac{x_3}{x_1} & \dots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \gamma \frac{x_3}{x_2} & \dots & \frac{x_{n-1}}{x_2} \\ \frac{1}{x_3} & \frac{x_1}{x_3} & \frac{x_2}{\gamma x_3} & 1 & \dots & \frac{x_{n-1}}{x_3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \frac{x_3}{x_{n-1}} & \dots & 1 \end{pmatrix}$$

Double perturbed PCM: $\mathbf{w}^{EM}$ is efficient

Theorem (Ábele-Nagy, Bozóki, Rebák, 2015):
The principal right eigenvector of a double perturbed pairwise comparison matrix is efficient.

Proof is based on the explicit formulas of $\mathbf{w}^{EM}$ and the characterization of efficiency by a strongly connected digraph.

$$
\mathbf{A}(p,q) = \begin{pmatrix}
1 & p & p & p & \dots & p & p \\
1/p & 1 & q & 1 & \dots & 1 & 1/q \\
1/p & 1/q & 1 & q & \dots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
1/p & 1 & 1 & 1 & \dots & 1 & q \\
1/p & q & 1 & 1 & \dots & 1/q & 1
\end{pmatrix},
$$

Proposition. (Bozóki, 2014) Let q be positive and $q \neq 1$. Then $\mathbf{w}^{EM}$ is internally inefficient, therefore inefficient. Furthermore, CR inconsistency can be arbitrarily small if q is close enough to 1.

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 1/9 & 9 & 1/9 & 1/8 \\ 1/4 & 1 & 1/8 & 1/4 & 1/7 & 1/5 \\ 9 & 8 & 1 & 8 & 4 & 1/2 \\ 1/9 & 4 & 1/8 & 1 & 7 & 1/3 \\ 9 & 7 & 1/4 & 1/7 & 1 & 1/5 \\ 8 & 5 & 2 & 3 & 5 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.1281 \\ 0.0180 \\ 0.3028 \\ 0.1237 \\ 0.1440 \\ 0.2835 \end{pmatrix}, \mathbf{w}^* = \begin{pmatrix} 0.1281 \\ \mathbf{0.0206} \\ \mathbf{0.3471} \\ 0.1237 \\ 0.1440 \\ \mathbf{0.3249} \end{pmatrix}.$$

Approximations are

$$\mathbf{X}^{EM} \qquad\qquad \mathbf{X}^*$$

$$\begin{pmatrix} 1 & 7.13 & 0.42 & 1.03 & 0.88 & 0.45 \\ 0.14 & 1 & 0.05 & 0.14 & 0.12 & 0.06 \\ 2.36 & 16.86 & 1 & 2.44 & 2.10 & 1.06 \\ 0.96 & 6.88 & 0.40 & 1 & 0.85 & 0.43 \\ 1.12 & 8.02 & 0.47 & 1.16 & 1 & 0.50 \\ 2.21 & 15.78 & 0.93 & 2.29 & 1.96 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{6.22} & \mathbf{0.36} & 1.03 & 0.88 & \mathbf{0.39} \\ \mathbf{0.16} & 1 & 0.05 & \mathbf{0.16} & \mathbf{0.14} & 0.06 \\ \mathbf{2.71} & 16.86 & 1 & \mathbf{2.80} & \mathbf{2.40} & 1.06 \\ 0.96 & \mathbf{6.01} & \mathbf{0.35} & 1 & 0.85 & \mathbf{0.38} \\ 1.12 & \mathbf{7.00} & \mathbf{0.41} & 1.16 & 1 & \mathbf{0.44} \\ \mathbf{2.53} & 15.78 & 0.93 & \mathbf{2.62} & \mathbf{2.25} & 1 \end{pmatrix}$$

Fichtner' metric

Theorem (Fichtner, 1984) Let $d : \mathcal{PCM}_n \times \mathcal{PCM}_n \to \mathbb{R}$ be as follows:

$$d(\mathbf{A}, \mathbf{B}) \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} \left(w_i^{EM(\mathbf{A})} - w_i^{EM(\mathbf{B})} \right)^2} + \frac{|\lambda_{\max}(\mathbf{A}) - \lambda_{\max}(\mathbf{B})|}{2(n-1)} + \\ + \chi(\mathbf{A}, \mathbf{B}) \frac{|\lambda_{\max}(\mathbf{A}) + \lambda_{\max}(\mathbf{B}) - 2n|}{2(n-1)},$$

where

$$\chi(\mathbf{A}, \mathbf{B}) = \begin{cases} 0 & \text{if } \mathbf{A} = \mathbf{B}, \\ 1 & \text{if } \mathbf{A} \neq \mathbf{B}. \end{cases}$$

Then, d is a metric in $\mathcal{PCM}_n$ with the following properties:

Fichtner' metric

(a) for every $\mathbf{A} \in \mathcal{PCM}_n$, $\mathbf{X}^{EM(\mathbf{A})}$ is the optimal solution of the problem $\min\{d(\mathbf{A},\mathbf{X})|\mathbf{X}$ is consistent$\}$;

(b)
$\min\{d(\mathbf{A},\mathbf{X})|\mathbf{X}$ is consistent$\} = d(\mathbf{A},\mathbf{X}^{EM(\mathbf{A})}) = \frac{\lambda_{\max}(\mathbf{A})-n}{n-1}$.

Optimality with respect to a nice objective function does not exclude inefficiency.

Note that Fichtner's metric is not continuous, nor a monotonic increasing function of $\left|a_{ij} - \frac{x_i}{x_j}\right|$.

Main references 1/2

Blanquero, R., Carrizosa, E., Conde, E. (2006): Inferring efficient weights from pairwise comparison matrices, *Mathematical Methods of Operations Research* **64**(2):271–284

Conde, E., Pérez, M.d.l.P.R. (2010): A linear optimization problem to derive relative weights using an interval judgement matrix, *European Journal of Operational Research* **201**(2):537–544

Fichtner, J. (1984): Some thoughts about the Mathematics of the Analytic Hierarchy Process, *Report 8403,* Universität der Bundeswehr München, Fakultät für Informatik, Institut für Angewandte Systemforschung und Operations Research, Werner-Heisenberg-Weg 39, D-8014 Neubiberg, F.R.G.

Main references 2/2

Ábele-Nagy, K., Bozóki, S. (2015): Efficiency analysis of simple perturbed pairwise comparison matrices, *under review*.

Ábele-Nagy, K., Bozóki, S., Rebák, Ö. (2015): Efficiency analysis of double perturbed pairwise comparison matrices, *manuscript*.

Bozóki, S. (2014): Inefficient weights from pairwise comparison matrices with arbitrarily small inconsistency, *Optimization*, **63**(12):1893–1901.

Bozóki, Fülöp, J. (2015): Efficient weight vectors from pairwise comparison matrices, *manuscript*.

Thank you for attention.

bozoki.sandor@sztaki.mta.hu

http://www.sztaki.mta.hu/∼bozoki