Incomplete pairwise comparison matrices

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29 June, 2010

Outline

- Pairwise comparison matrix
- Incomplete pairwise comparison matrix
- LSM method for incomplete matrices
- A conjecture

Given n objects with weights $w_1, w_2, w_3, \ldots, w_n$. The pairwise comparison matrix is defined as follows:

$$\begin{pmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \ldots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \frac{w_2}{w_3} & \ldots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & 1 & \ldots & \frac{w_3}{w_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \ldots & 1 \end{pmatrix},$$

where

$$\begin{aligned} w_{ij} &> 0, \\ w_{ij} &= \frac{1}{w_{ji}}, \\ w_{ij} &= w_{ik} w_{kj}. \end{aligned}$$

for any i, j, k indices.

In real decision situations, weights are unknown, but pairwise comparisons can be made:

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix},$$

where

$$\begin{aligned} a_{ij} &> 0, \\ a_{ij} &= \frac{1}{a_{ji}}. \end{aligned}$$

for $i, j = 1, \dots, n$. The aim is to determine the weight vector $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n_+$.

Least Squares Method, LSM

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left[a_{ij} - \left(\frac{w_i}{w_j} \right) \right]^2$$

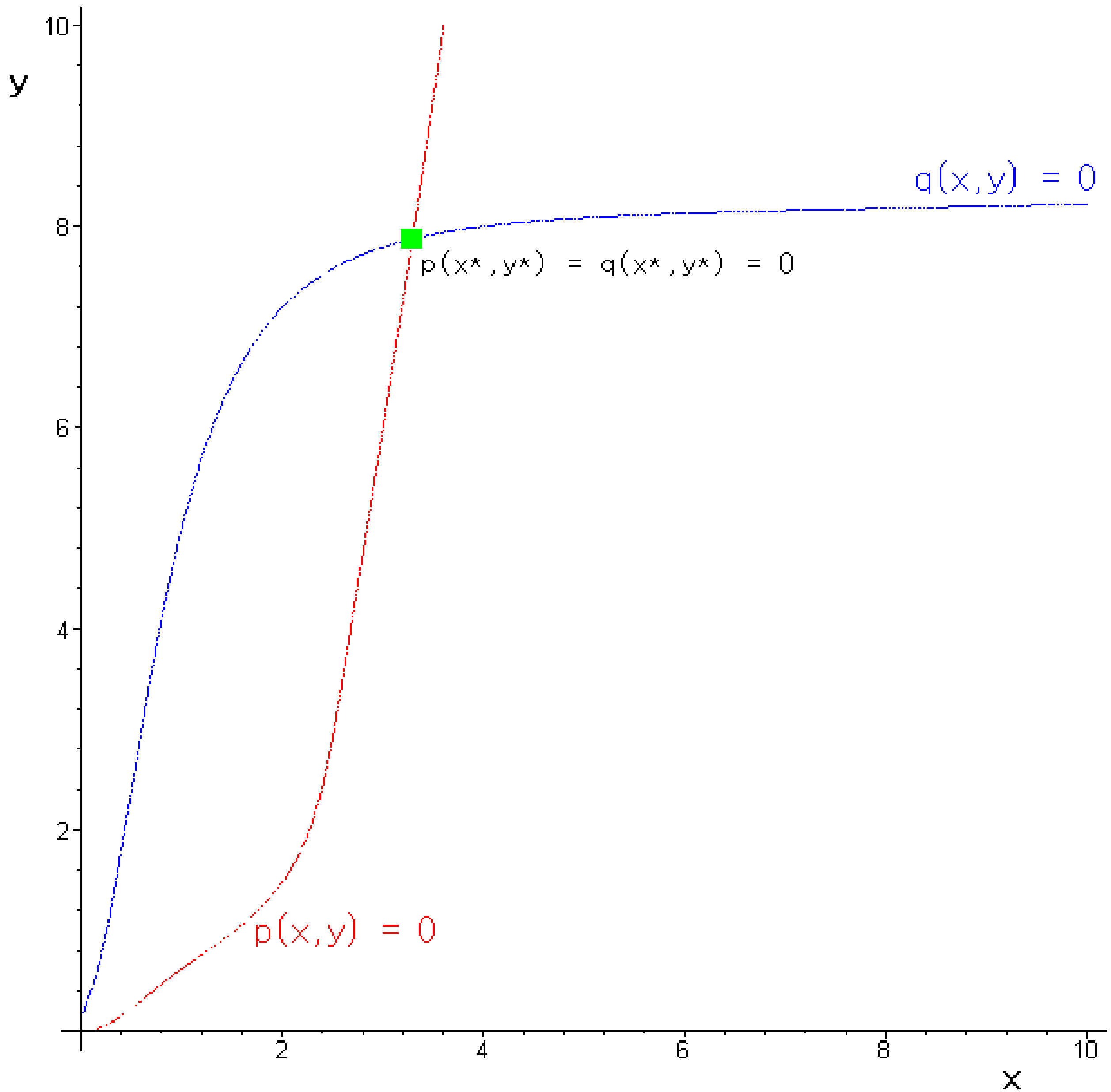
$$\sum_{i=1}^{n} w_i = 1,$$

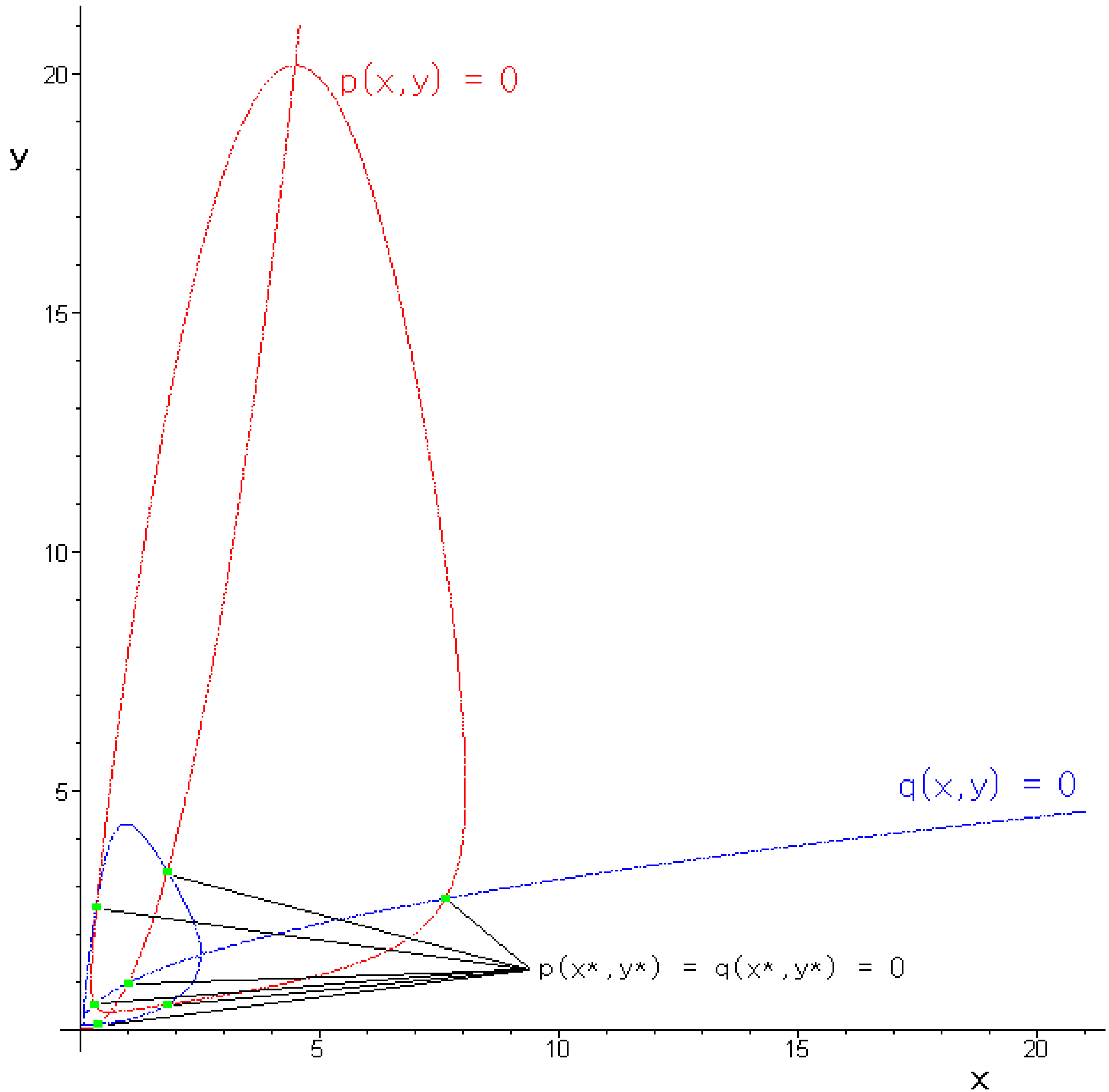
$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

First order optimality conditions lead to a multivariate polynomial system.

Solution of the polynomial system:

- resultant method (for 3×3 matrices)
- generalized resultant method (for 3×3 and 4×4 matrices) The code written by Robert H. Lewis (Fordham Univ.) in Fermat was used.
- homotopy method (for $3 \times 3, 4 \times 4, \ldots, 8 \times 8$ matrices) The code *hom4ps* written by Tien-Yien Li (Michigan State Univ.) and Tangan Gao (California State Univ.) was used.


y
10
8
6
4
2
q(x,y) = 0
p(x*,y*) = q(x*,y*) = 0
p(x,y) = 0
2
4
6
8
10
x


p(x,y) = 0
q(x,y) = 0
p(x*,y*) = q(x*,y*) = 0
y
x
20
15
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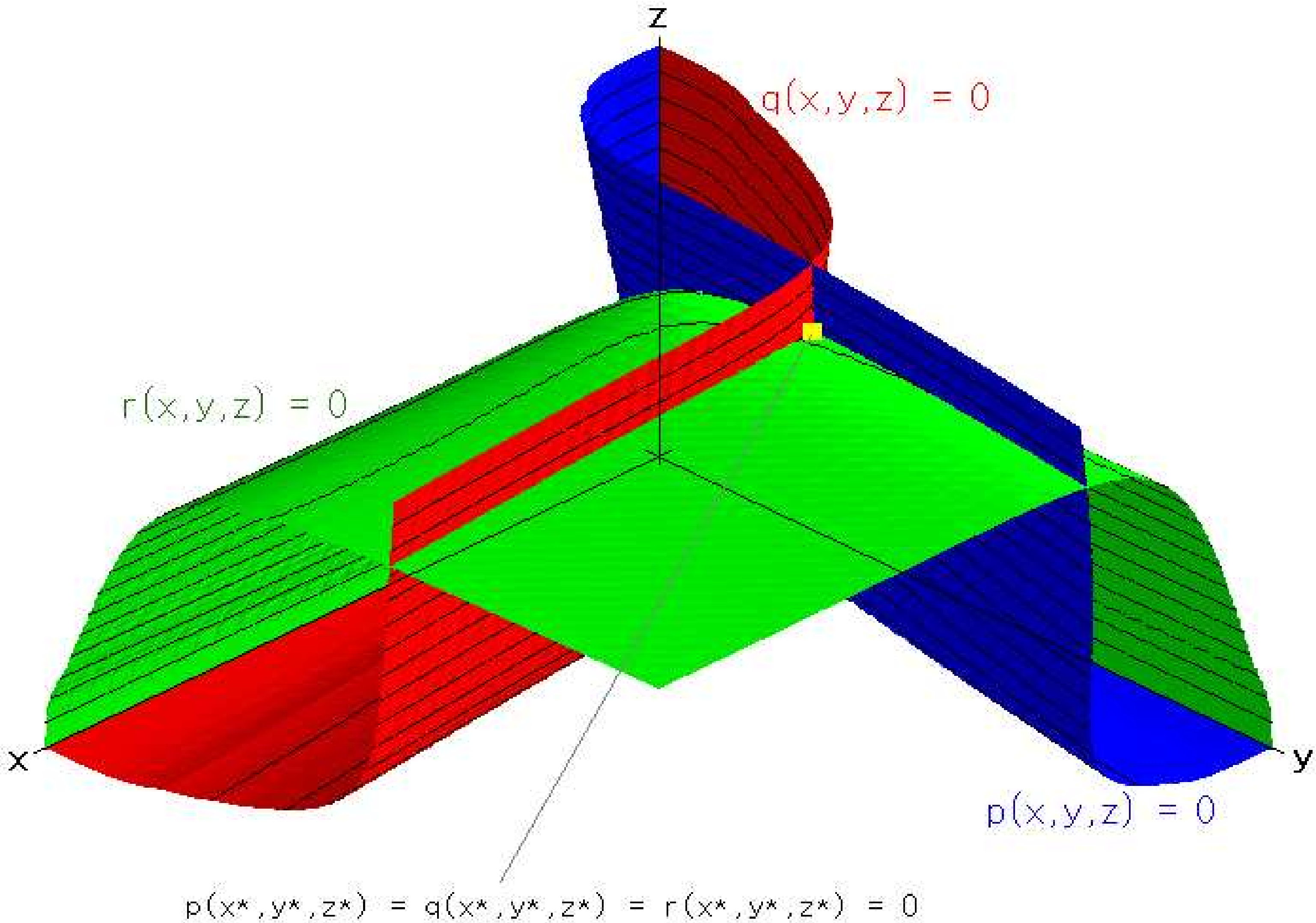
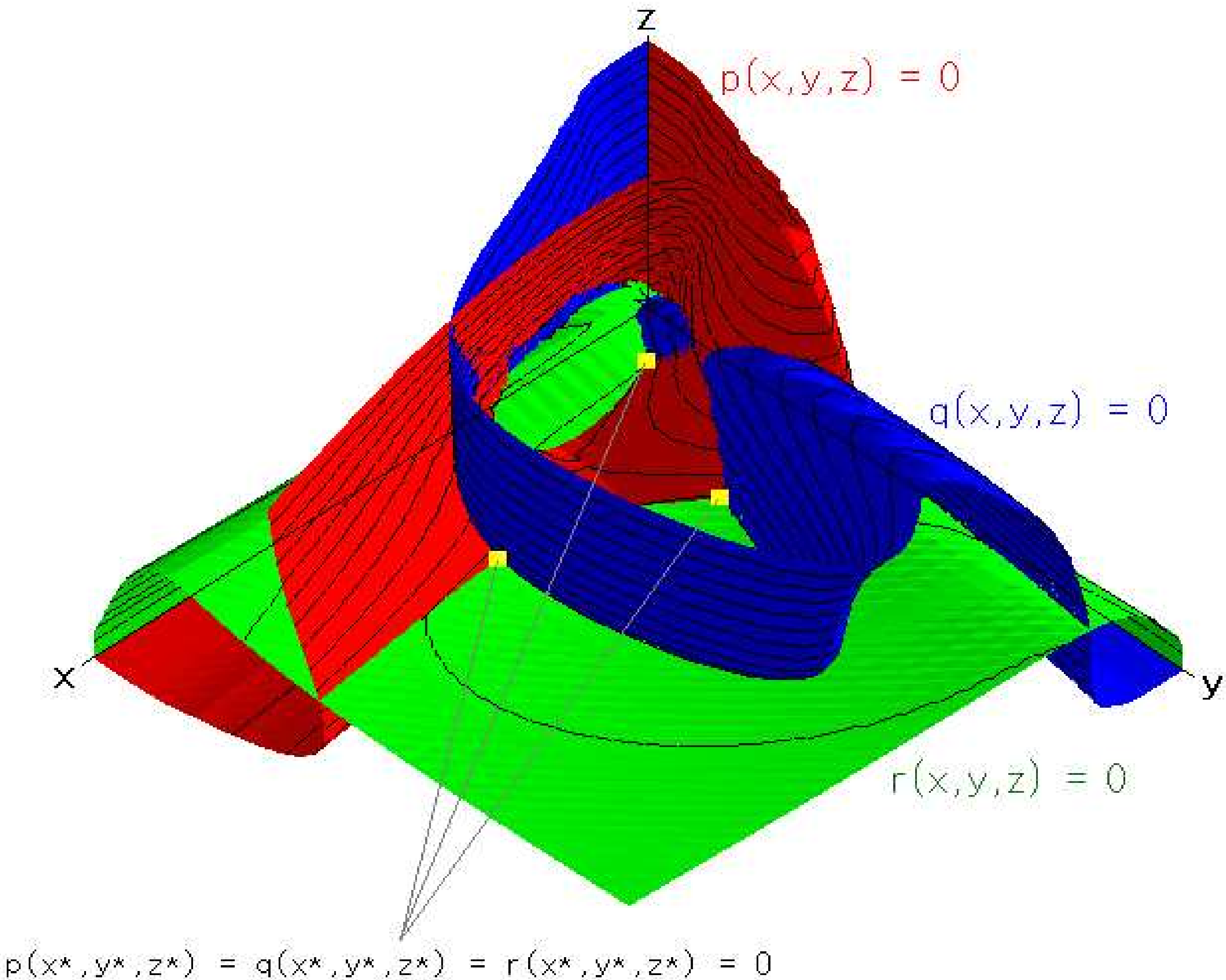
20


z
q(x,y,z) = 0
r(x,y,z) = 0
x
y
p(x,y,z) = 0
p(x*,y*,z*) = q(x*,y*,z*) = r(x*,y*,z*) = 0


z
p(x,y,z) = 0
q(x,y,z) = 0
x
y
r(x,y,z) = 0
p(x*,y*,z*) = q(x*,y*,z*) = r(x*,y*,z*) = 0

Size of matrix ($n \times n$)	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
CPU time	0.05 sec	0.5 sec	20 sec	14 min	10 hours	3 days
Number of common roots	24	224	1640	$O(10^4)$	$O(10^5)$	$O(10^6)$
Number of real common roots	4–10	8–18	16–46	32–76	64–92	128–160
Number of positive real common roots	1–7	1–11	1–31	1–15	1–28	1–21
Number of local minima	0–1	0–3	0–5	0–5	0–7	0–7
Number of global minima	1–3	1–4	1–5	1–6	1–7	1–8
Total number of minima	1–4	1–4	1–10	1–6	1–14	1–8

Pairwise comparison matrix with missing elements (incomplete pairwise comparison matrix)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & - & \ldots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \ldots & - \\ - & 1/a_{23} & 1 & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & - & 1/a_{3n} & \ldots & 1 \end{pmatrix}.$$

Pairwise comparison matrix with missing elements (incomplete pairwise comparison matrix)

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Pairwise comparison matrix with missing elements (incomplete pairwise comparison matrix)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & - \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & - & 1/a_{3n} & \dots & 1 \end{pmatrix}.$$

Pairwise comparison matrix with missing elements (incomplete pairwise comparison matrix)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & x_d \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/x_d & 1/a_{3n} & \dots & 1 \end{pmatrix},$$

Pairwise comparison matrix with missing elements (incomplete pairwise comparison matrix)

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & x_1 & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & x_d \\ 1/x_1 & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/x_d & 1/a_{3n} & \dots & 1 \end{pmatrix},$$

where $x_1, x_2, \dots, x_d \in \mathbb{R}_+$.

Graph representation of a pairwise comparison matrix

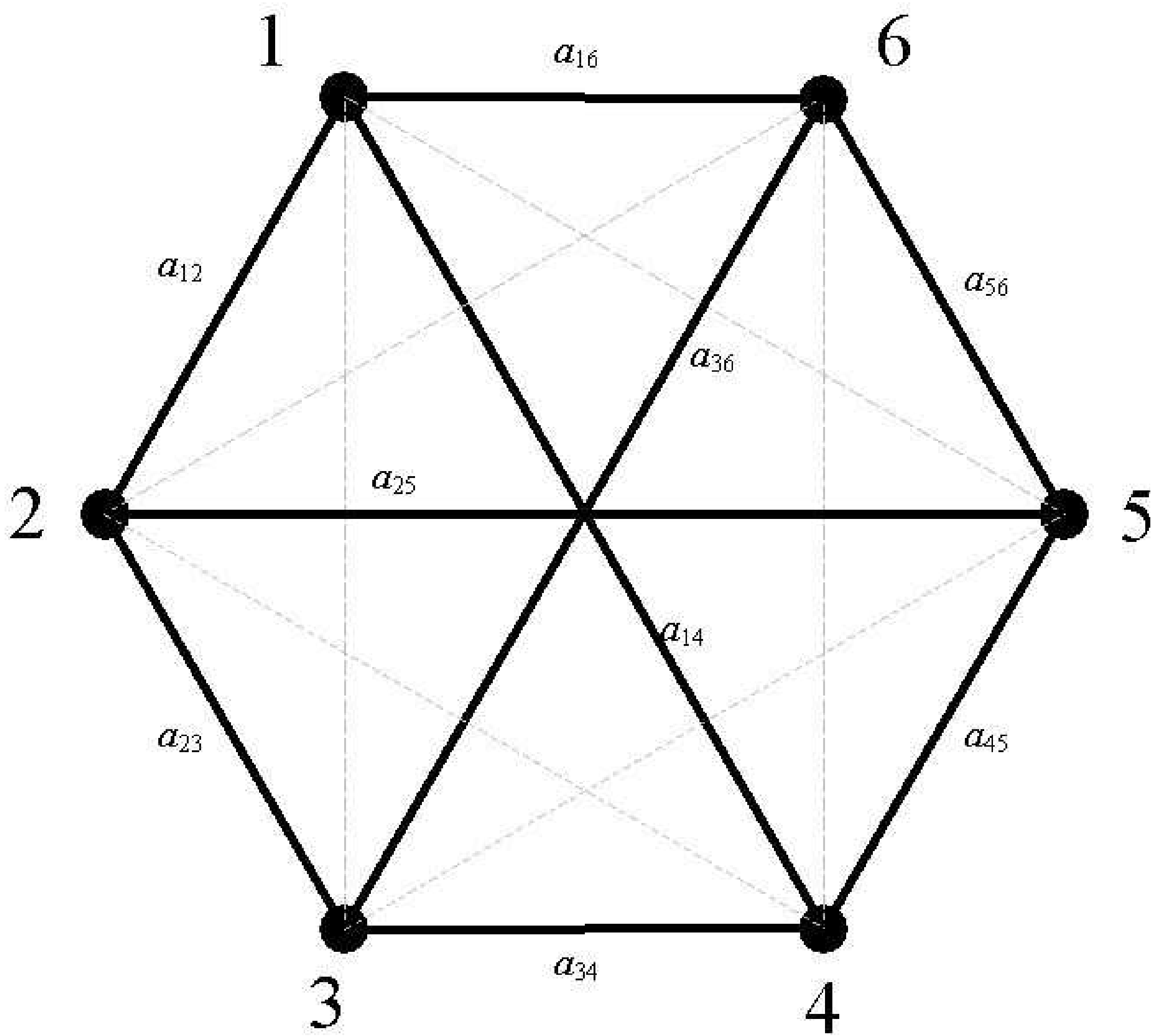
Given $\mathbf{A}$ incomplete pairwise comparison matrix of size $n \times n$. Graph $G = (V, E)$ is defined as follows:

$$V = \{1, 2, \ldots, n\}$$

$E = \{e(i,j) \,|\, a_{ij}$ (and a_{ji}) are given and $i \neq j\}$

Special case: all the comparisons are given, the corresponding graph is K_n.

Graph representation of a pairwise comparison matrix



	1	2	3	4	5	6
1	-	a_{12}	-	a_{14}	-	a_{16}
2	a_{21}	-	a_{23}	-	a_{25}	-
3	-	a_{32}	-	a_{34}	-	a_{36}
4	a_{41}	-	a_{43}	-	a_{45}	-
5	-	a_{52}	-	a_{54}	-	a_{56}
6	a_{61}	-	a_{63}	-	a_{65}	-

LSM problem for incomplete matrices:

$$\min \sum_{\substack{(i,j) \in E \\ 1 \leq i < j \leq n}} \left[a_{ij} - \left(\frac{w_i}{w_j} \right) \right]^2$$

$$\sum_{i=1}^{n} w_i = 1,$$

$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

$$\min \sum_{\substack{(i,j) \in E \\ 1 \le i < j \le n}} \left[a_{ij} - \left(\frac{w_i}{w_j} \right) \right]^2$$

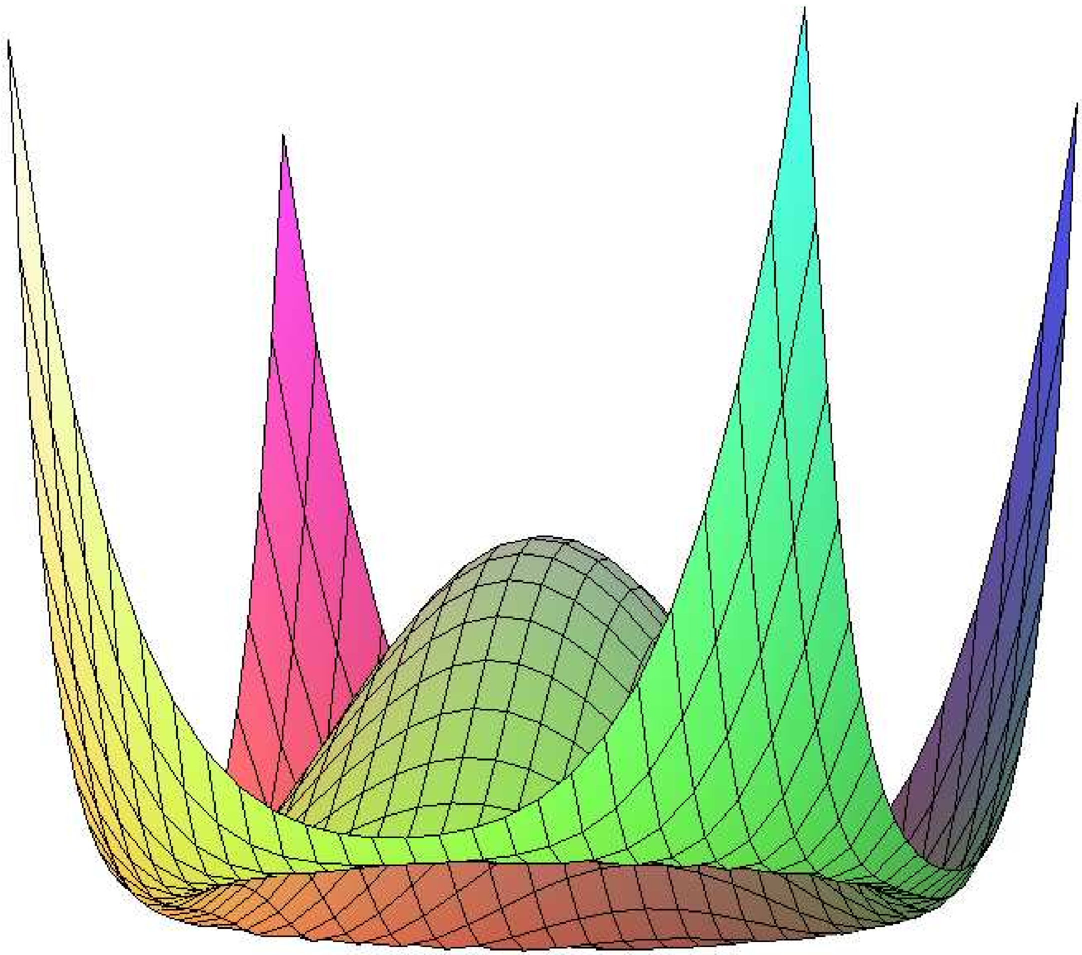
$$\sum_{i=1}^{n} w_i = 1,$$

$$w_i > 0, \qquad i = 1, 2, \dots, n.$$

Conjecture: The set of optimal solutions of the incomplete LSM problem above is *finite* if and only if graph G corresponding to the incomplete pairwise comparison matrix is *connected*.

Any ideas or suggestions regarding the proof or disproof of the conjecture are welcome.

Example: function $(x^2 + y^2)^2 - 40(x^2 + y^2)$ has infinite number of minima.



Thank you for attention.

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