Nontransitive dice

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Outline

- Nontransitive dice
- Quadratic residues mod $p = 4m + 3$
- Schütte property (S_k) and Paley tournament (P_p)
- Nontransitive dice set with property S_k

Nontransitive dice

R \ G	2	2	2	5	5	5
1	2	2	2	5	5	5
4	4	4	4	5	5	5
4	4	4	4	5	5	5
4	4	4	4	5	5	5
4	4	4	4	5	5	5
4	4	4	4	5	5	5

15 < 21

G \ B	3	3	3	3	3	6
2	3	3	3	3	3	6
2	3	3	3	3	3	6
2	3	3	3	3	3	6
5	5	5	5	5	5	6
5	5	5	5	5	5	6
5	5	5	5	5	5	6

15 < 21

B \ R	1	4	4	4	4	4
3	3	4	4	4	4	4
3	3	4	4	4	4	4
3	3	4	4	4	4	4
3	3	4	4	4	4	4
3	3	4	4	4	4	4
6	6	6	6	6	6	6

11 < 25

Binary relation $X\mathcal{R}Y \Leftrightarrow prob(X > Y) > \frac{1}{2}$ is not transitive.

Early history

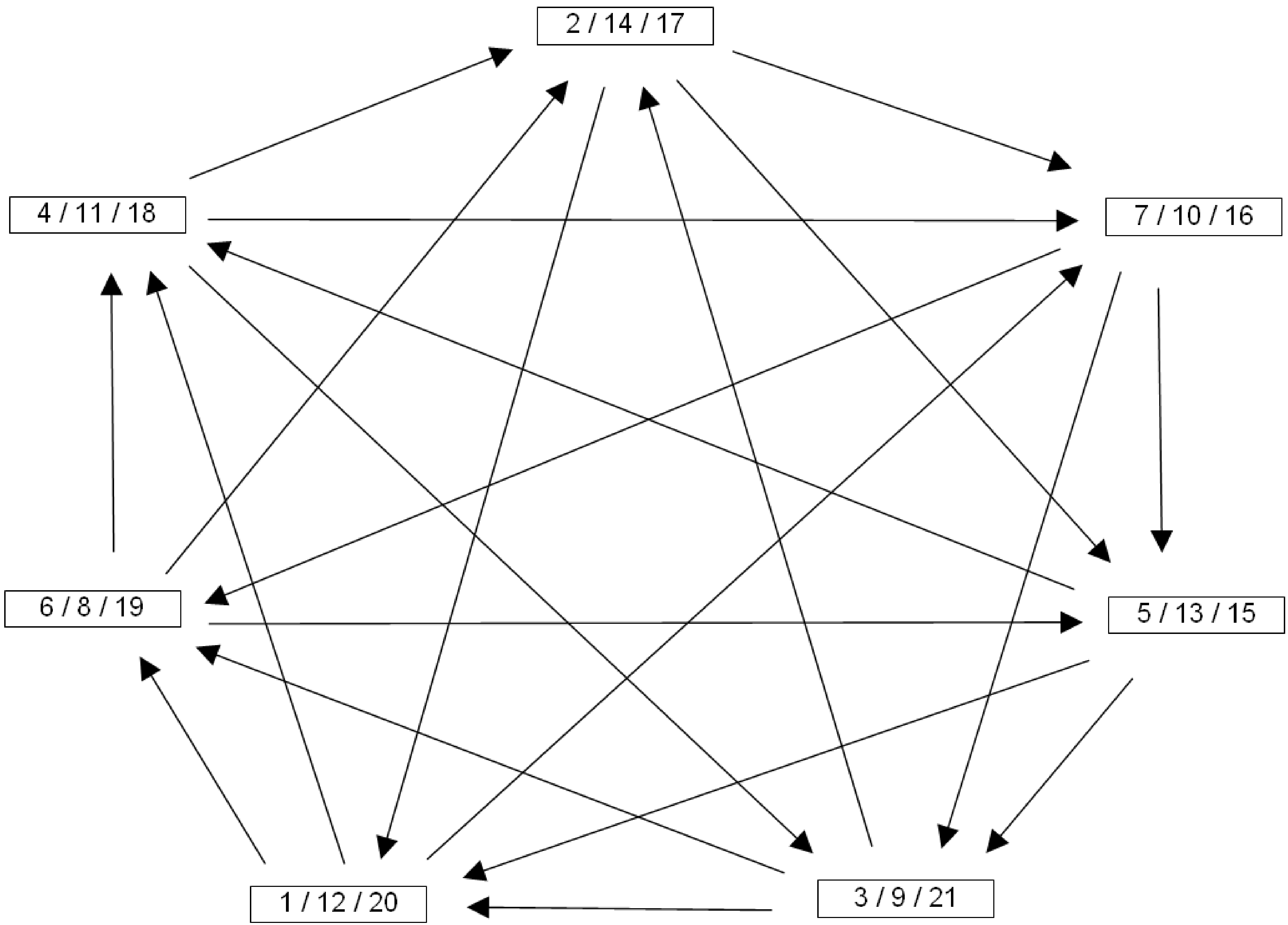
Steinhaus and Trybula (1959)

Usiskin (1964)

Moon and Moser (1967)

Efron (1970)

van Deventer's dice set for three players



Dice set for n players?

We are looking for dice sets for n players, that is, whatever the first $n-1$ opponents choose, the n-th player will find (at least) one of the remaining dice that beats all opponents' dice.

Quadratic residues

Let $p = 4m + 3$ be a prime number.
q is called a quadratic residue modulo p, if q is congruent to a perfect square modulo p, otherwise it is called a quadratic nonresidue.

Theorem (Dirichlet, 1839) *For $p \geq 7$, the sum of the residues minus the sum of the nonresidues is a negative number, and it is an odd multiple of p.*

For $p = 7$, the sum of quadratic residues is $(0+)1 + 2 + 4 = 7$, while the sum of quadratic nonresidues is $3 + 5 + 6 = 14$, and their difference is $7 - 14 = -1 \cdot 7$.

Schütte property

Erdős writes in 1963:

> „*The problem was recently put to me by Professor Schütte in its graph-theoretic form: If* $\mathcal{G}^{(n)}$ *is a complete directed graph, with* n *vertices, which has the property that for every* k *vertices of* $\mathcal{G}^{(n)}$ *there is at least one vertex from which edges go out to each of the* k*, we shall say that* $\mathcal{G}^{(n)}$ *has the property* S_k*. Schütte's problem is to show that for every* k *there is a* $\mathcal{G}^{(n)}$ *with the property* S_k *and to find the least possible* n *for a given* k*.*"

Schütte property

That $f(k) < \infty$ (for all $k = 1, 2, \ldots$) is proved first by Erdős (1963). Lower bounds for $f(k)$:

$$f(k) \geq 2^{k+1} - 1 \quad \text{(Erdős, 1963)}$$

$$f(k) \geq 2^{k-1}(k+2) - 1 \quad \text{(Szekeres and Szekeres, 1965)}$$

$$\limsup_{k} \frac{f(k)}{2^k k^2} \geq \log 2 \quad \text{(Erdős, 1963)}$$

However, exact values of $f(k)$ are known only for small k. Szekeres and Szekeres (1965) showed that $f(1) = 3,\ f(2) = 7,\ f(3) = 19.$

Paley tournament

Let $p = 4m + 3$ be a prime number. *Paley tournament* on p vertices, denoted by P_p, is defined by its incidence matrix $\mathbf{M}$ of size $p \times p$ as follows:

$$\mathbf{M}_{i,j} := \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i - j \text{ is a quadratic residue modulo } p \\ -1 & \text{if } j - i \text{ is a quadratic residue modulo } p \end{cases}$$

Paley tournaments with property S_k

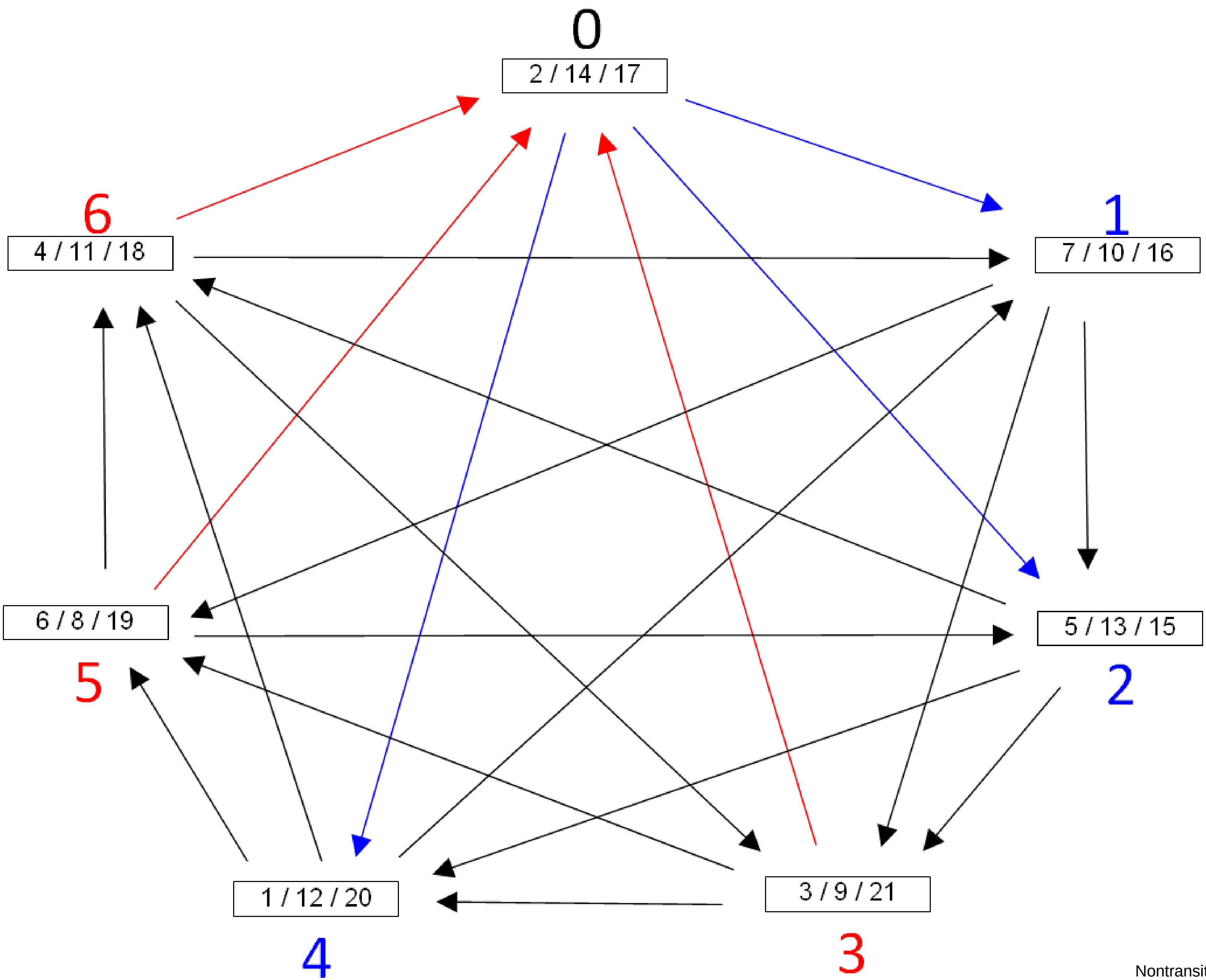
Upper bound for $f(k)$:

Theorem (Graham and Spencer) *If $p > k^2 2^{2k-2}$, then P_p has property S_k.*

According to Reid, McRae, Hedetniemi and Hedetniemi (2004) it was computationally verified by Fisher in 1996 that

Among Paley tournaments:
(a) *P_{67} is the smallest one having property S_4*
(b) *P_{331} is the smallest one having property S_5*
(c) *P_{1163} is the smallest one having property S_6.*

van Deventer's dice set for three players



van Deventer's dice set

	die 0	die 1	die 2	die 3	die 4	die 5	die 6	add
face 1	0	1	2	3	4	5	6	$+1+0\cdot 7$
face 2	4	6	1	3	5	0	2	$+1+1\cdot 7$
face 3	5	2	6	3	0	4	1	$+1+2\cdot 7$

Dice set for n players

A construction of dice set $\mathbf{D}_p$ of p dice is sketched, then it is shown that it realizes P_p.

Remember that $p = 4m + 3$ is a prime number. Each die has $p(p-1)/2$ faces having equal probabilities.

The dice set is written as an array of $p(p-1)/2$ rows and p columns. Each column is associated with a die and faces are organized in rows.

Quadratic residues modulo p are denoted by $q_0 = 0, q_1 = 1, q_2, q_3, \ldots, q_{(p-1)/2}$ as before. Since $q_1 = 1$ for any p, the first $p \times p$ array can be written as in table as follows.

The first $p \times p$ array of dice set D_p generated by $q_1 = 1$

	die 0	die 1	die 2	...	die $p-1$	add
face 1	**0**	**1**	**2**	...	$\boldsymbol{p-1}$	$+\,0 \cdot p$
face 2	1	2	3	...	0	$+1 \cdot p$
face 3	2	3	4	...	1	$+2 \cdot p$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
face p	$p-1$	0	1	...	$p-2$	$+(p-1) \cdot p$

The m-th $p \times p$ array of dice set D_p generated by q_m

	die 0	die 1	die 2	...	die $p-1$	add
face ($m-1$)p+1	**d_0=0**	**...**	**...**	**...**	**...**	$+(m-1)p^2$
face $(m-1)p+2$	1	...	...	...	...	$+(m-1)p^2+p$
face $(m-1)p+3$	2	...	...	...	...	$+(m-1)p^2+2p$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
face mp	$p-1$	...	...	...	...	$+mp^2$

Dice set D_7

	die 0	die 1	die 2	die 3	die 4	die 5	die 6	add
face 1	**0**	**1**	**2**	**3**	**4**	**5**	**6**	$+0 \cdot 7$
face 2	1	2	3	4	5	6	0	$+1 \cdot 7$
face 3	2	3	4	5	6	0	1	$+2 \cdot 7$
face 4	3	4	5	6	0	1	2	$+3 \cdot 7$
face 5	4	5	6	0	1	2	3	$+4 \cdot 7$
face 6	5	6	0	1	2	3	4	$+5 \cdot 7$
face 7	6	0	1	2	3	4	5	$+6 \cdot 7$

Dice set D_7

	die 0	die 1	die 2	die 3	die 4	die 5	die 6	add
face 8	**0**	**4**	**1**	**5**	**2**	**6**	**3**	$+7 \cdot 7$
face 9	1	5	2	6	3	0	4	$+8 \cdot 7$
face 10	2	6	3	0	4	1	5	$+9 \cdot 7$
face 11	3	0	4	1	5	2	6	$+10 \cdot 7$
face 12	4	1	5	2	6	3	0	$+11 \cdot 7$
face 13	5	2	6	3	0	4	1	$+12 \cdot 7$
face 14	6	3	0	4	1	5	2	$+13 \cdot 7$

Dice set D_7

	die 0	die 1	die 2	die 3	die 4	die 5	die 6	add
face 15	**0**	**2**	**4**	**6**	**1**	**3**	**5**	+14·7
face 16	1	3	5	0	2	4	6	+15·7
face 17	2	4	6	1	3	5	0	+16·7
face 18	3	5	0	2	4	6	1	+17·7
face 19	4	6	1	3	5	0	2	+18·7
face 20	5	0	2	4	6	1	3	+19·7
face 21	6	1	3	5	0	2	4	+20·7

Proposition. *Dice set* D_p *realizes* P_p.

Proof is built up from four lemmas.

Lemma 1. *The probability that die* 0 *beats die* i *is smaller than, equal to or larger than* $1/2$ *if and only if the respective probability that die* 0 *beats die* i*, restricted to the cases where their same-indexed faces compete to each other, is smaller than, equal to or larger than* $1/2$.

Lemma 2. *The odds of die 0 vs. die i, restricted to the cases when their same-indexed faces compete against each other, are equal to* $\left[\sum_{\ell=1}^{(p-1)/2} v_{\ell,i}\right] : \left[\sum_{\ell=1}^{(p-1)/2} (p - v_{\ell,i})\right]$, *where $v_{\ell,i}$ denotes the value written on ℓ-th key face of die i, and the sums are taken over the $(p-1)/2$ key faces.*

Lemma 3. *Nonzero quadratic residues are written on key faces of die i, where i is a nonzero quadratic residue. Equivalently, quadratic nonresidues are written on key faces of dice j, where j is from the set of quadratic nonresidues.*

Lemma 4. *The probability that die 0 beats die i is greater than $1/2$ if and only if i is a quadratic nonresidue. The probability that die 0 beats die i is smaller than $1/2$ if and only if i is a nonzero quadratic residue.*

Proposition *If $p > k^2 2^{2k-2}$, then D_p is a nontransitive dice set for $k+1$ players.*

Conclusion

Finite dice sets with Schütte property have been proposed: for an arbitrary subset of k dice there is at least one die that beats each of the k with a probability greater than $1/2$.

Results are published in

Bozóki, S. [2014]: Nontransitive dice sets realizing the Paley tournaments for solving Schütte's tournament problem, *Miskolc Mathematical Notes*, 15(1), pp.39-50.
http://mat76.mat.uni-miskolc.hu/%7Emnotes/downloader.php?article=mmn_659.pdf

Thank you for attention.

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