Pairwise comparison matrices and efficient weight vectors

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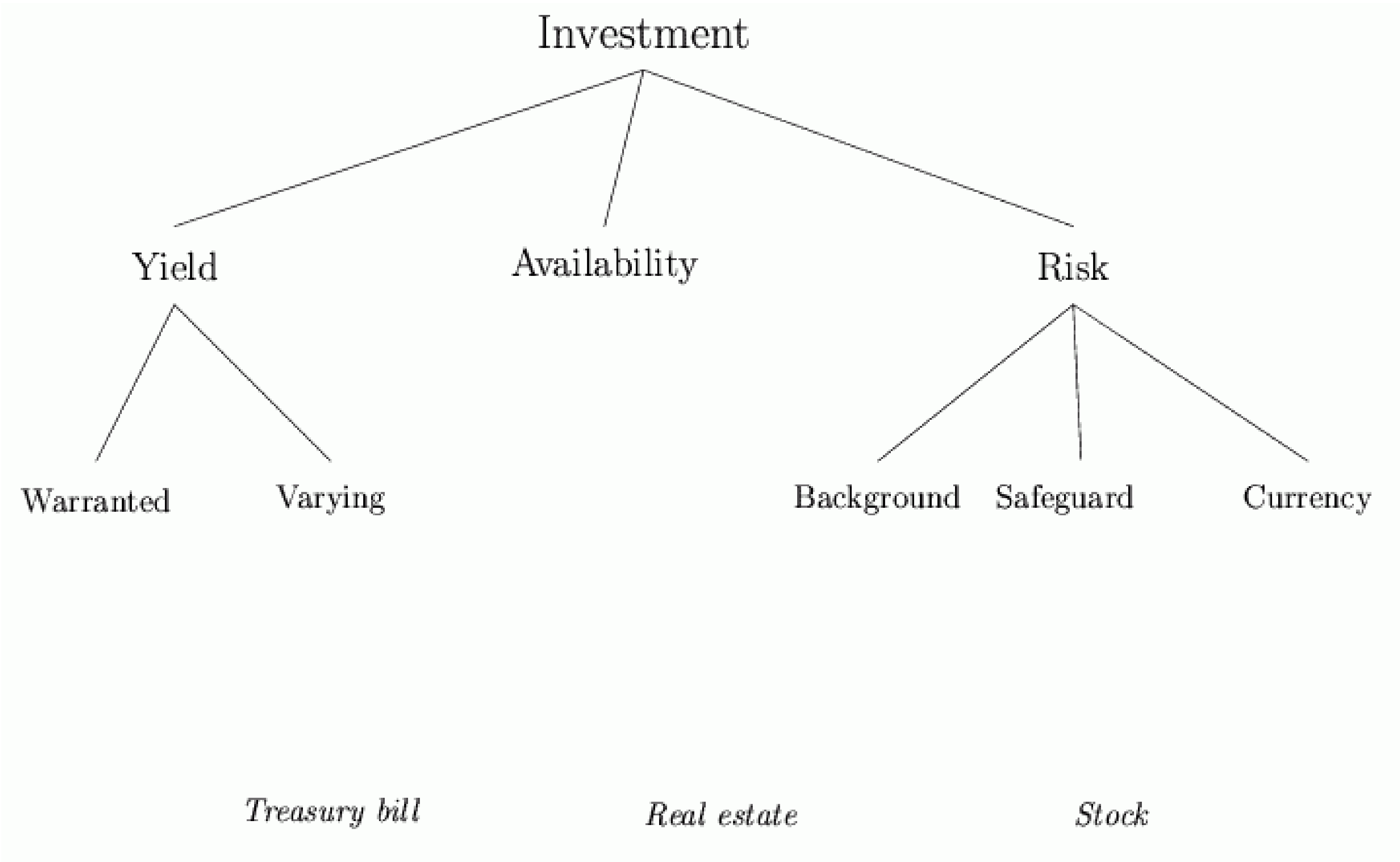
Multi Criteria Decision Making

Analytic Hierarchy Process

Criterion tree

Pairwise comparison matrix

Criterion tree



Pairwise comparison matrix

In practical problems

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix},$$

is given, where for any $i, j = 1, \dots, n$ indices

$$a_{ij} > 0, \qquad a_{ij} = \frac{1}{a_{ji}}.$$

The aim is to find the $\mathbf{w} = (w_1, w_2, \dots, w_n)^\top \in \mathbb{R}^n_+$ weight vector.

Weighting methods

Eigenvector Method (Saaty): $\mathbf{A}\mathbf{w} = \lambda_{max}\mathbf{w}.$

Least Squares Method (LSM):

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - \frac{w_i}{w_j} \right)^2$$

$$\sum_{i=1}^{n} w_i = 1, \qquad w_i > 0, \quad i = 1, 2, \ldots, n.$$

Logarithmic Least Squares Method (LLSM):

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\log a_{ij} - \log \frac{w_i}{w_j} \right)^2$$

$$\sum_{i=1}^{n} w_i = 1, \qquad w_i > 0, \quad i = 1, 2, \ldots, n.$$

$$\begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix},$$

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$$\left[\frac{w_i^{EM}}{w_j^{EM}}\right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}$$

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$$\left[\frac{w_i'}{w_j'}\right] = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{3.9546} & \mathbf{8.8989} \\ \mathbf{1} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2529} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1124} & 0.1124 & 0.4444 & 1 \end{pmatrix}.$$

The multi-objective optimization problem is as follows:

$$\min_{x_i > 0 \ \forall i} \left(\left| a_{ij} - \frac{x_i}{x_j} \right| \right)_{i \neq j}$$

Efficiency (Pareto optimality)

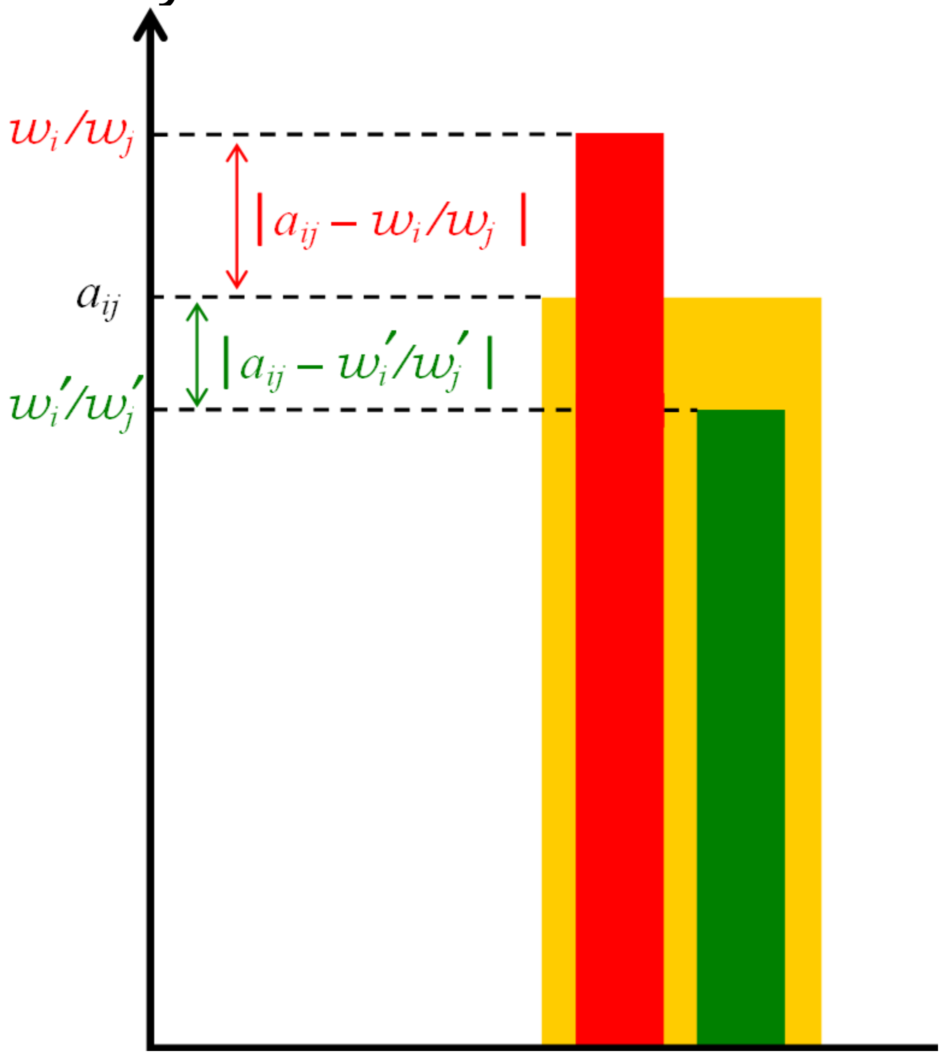
Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n}$ be an $n \times n$ pairwise comparison matrix and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector.

Definition: weight vector $\mathbf{w}$ is called *efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \le \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \le i, j \le n,$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \qquad \text{for some } 1 \le k, \ell \le n.$$

An efficient weight vector cannot be improved such that every element of the pairwise comparison matrix is approximated at least as good, and at least one element is approximated strictly better.



Test of efficiency

Given pairwise comparison matrix $\mathbf{A}$ and weight vector $\mathbf{w}$, our goal is check whether $\mathbf{w}$ is efficient.

Let $v_i = \log w_i$, $1 \le i \le n$, and $b_{ij} = \log a_{ij}$, $1 \le i, j \le n$,

$$I = \left\{ (i,j) \,\middle|\, a_{ij} < \frac{w_i}{w_j} \right\}$$

$$J = \left\{ (i,j) \,\middle|\, a_{ij} = \frac{w_i}{w_j}, i < j \right\}$$

$$\min \sum_{(i,j) \in I} -s_{ij}$$

$$y_j - y_i \le -b_{ij} \qquad \text{for all } (i,j) \in I,$$

$$y_i - y_j + s_{ij} \le v_i - v_j \qquad \text{for all } (i,j) \in I,$$

$$y_i - y_j = b_{ij} \qquad \text{for all } (i,j) \in J,$$

$$s_{ij} \ge 0 \qquad \text{for all } (i,j) \in I,$$

$$y_1 = 0$$

Variables are y_i, $1 \le i \le n$ and $s_{ij} \ge 0$, $(i,j) \in I$.

$$\min \sum_{(i,j)\in I} -s_{ij}$$

$$y_j - y_i \leq -b_{ij} \qquad \text{for all } (i,j) \in I,$$

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$$s_{ij} \geq 0 \qquad \text{for all } (i,j) \in I,$$

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Theorem (Bozóki, Fülöp, 2016):
The optimum value of the linear program above is at most 0 and it is equal to 0 if and only if weight vector $\mathbf{w}$ is efficient. Denote the optimal solution to the LP above by $(\mathbf{y}^*, \mathbf{s}^*) \in \mathbb{R}^{n+|I|}$. If weight vector $\mathbf{w}$ is inefficient, then weight vector $\exp(\mathbf{y}^*)$ is efficient and dominates $\mathbf{w}$ internally.

Pairwise Comparison Matrix Calculator

The efficiency of a weight vector can be tested at

pcmc.online

If the weight vector is found to be inefficient, then a dominating efficient weight vector is provided.

PCMC deals with incomplete pairwise comparison matrices, too.

Characterization of efficiency

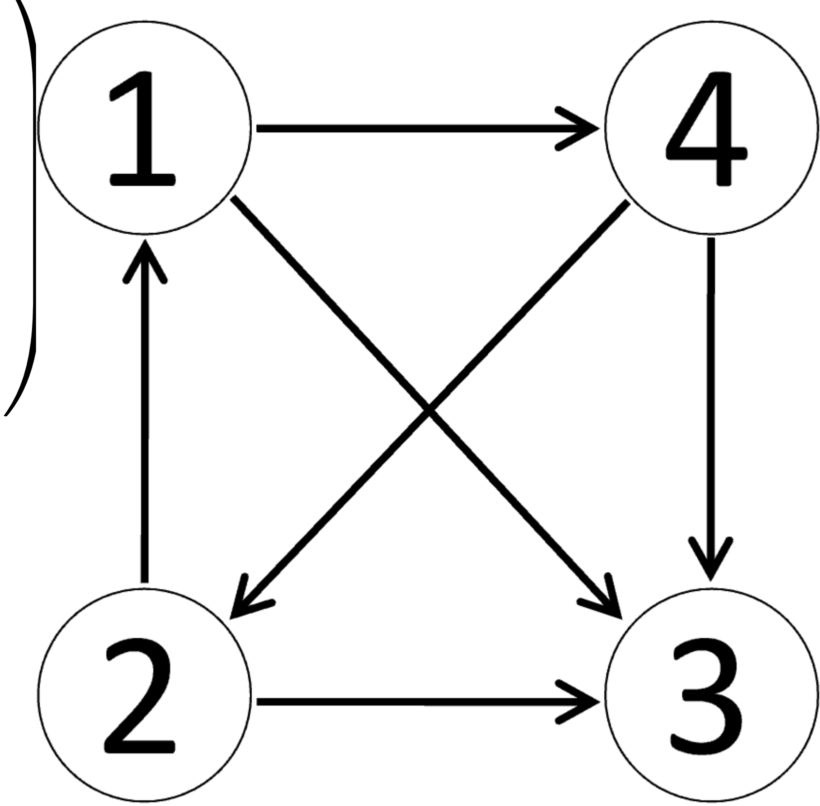
Definition: Let $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathcal{PCM}_n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector. Directed graph $(V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is defined as follows: $V = \{1, 2, \ldots, n\}$ and

$$\overrightarrow{E} = \left\{ \text{arc}(i \to j) \,\middle|\, \frac{w_i}{w_j} \geq a_{ij}, i \neq j \right\}.$$

Theorem (Blanquero, Carrizosa and Conde, 2006): Weight vector $\mathbf{w}$ is efficient if and only if $(V, \overrightarrow{E})_{\mathbf{A},\mathbf{w}}$ is strongly connected, that is, there exist directed paths from i to j and from j to i for all pairs of $i \neq j$ nodes.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 2 \\ 1/2 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/2 & 1/3 & 2 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1 \\ 2.0712416 \end{pmatrix}$$

$$\mathbf{X}^{EM} = \begin{pmatrix} 1 & \color{blue}{1.41} & \color{red}{6.01} & \color{red}{2.90} \\ \color{red}{0.71} & 1 & \color{red}{4.26} & \color{blue}{2.06} \\ \color{blue}{0.1663} & \color{blue}{0.23} & 1 & \color{blue}{0.48} \\ \color{blue}{0.34} & \color{red}{0.49} & \color{red}{2.07} & 1 \end{pmatrix}$$



Efficiency of the principal right eigenvector

Special cases

Efficient principal right eigenvector:

- simple perturbed PCM
- double perturbed PCM

Inefficient principal right eigenvector:

- PCM with arbitrarily small inconsistency
- Numerical examples

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Fichtner's metric

Theorem (Fichtner, 1984)
Let $d : \mathcal{PCM}_n \times \mathcal{PCM}_n \to \mathbb{R}$ be as follows:

$$d(\mathbf{A}, \mathbf{B}) \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} \left(w_i^{EM(\mathbf{A})} - w_i^{EM(\mathbf{B})} \right)^2} + \frac{|\lambda_{\max}(\mathbf{A}) - \lambda_{\max}(\mathbf{B})|}{2(n-1)} +$$

$$+\chi(\mathbf{A}, \mathbf{B}) \frac{|\lambda_{\max}(\mathbf{A}) + \lambda_{\max}(\mathbf{B}) - 2n|}{2(n-1)},$$

where

$$\chi(\mathbf{A}, \mathbf{B}) = \begin{cases} 0 & \text{if } \mathbf{A} = \mathbf{B}, \\ 1 & \text{if } \mathbf{A} \neq \mathbf{B}. \end{cases}$$

Then, d is a metric in $\mathcal{PCM}_n$ with the following properties:

Fichtner's metric

(a) for every $\mathbf{A} \in \mathcal{PCM}_n$, $\mathbf{X}^{EM(\mathbf{A})}$ is the optimal solution of the problem $\min\{d(\mathbf{A}, \mathbf{X}) | \mathbf{X}$ is consistent$\}$;

(b)
$\min\{d(\mathbf{A}, \mathbf{X}) | \mathbf{X}$ is consistent$\} = d(\mathbf{A}, \mathbf{X}^{EM(\mathbf{A})}) = \frac{\lambda_{\max}(\mathbf{A}) - n}{n-1}$.

Optimality with respect to a nice objective function does not exclude inefficiency.

Note that Fichtner's metric is not continuous, nor a monotonic increasing function of $\left| a_{ij} - \frac{x_i}{x_j} \right|$.

Simple perturbed PCM

Consider a consistent matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & x_1 & x_2 & \dots & x_{n-1} \\ \frac{1}{x_1} & 1 & \frac{x_2}{x_1} & \dots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \dots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \dots & 1 \end{pmatrix} \in \mathcal{PCM}_n,$$

then perturb a single element and its reciprocal. The perturbation is realized by a multiplication by $\delta > 0, \delta \neq 1$, while the reciprocal element is divided by δ.

Simple perturbed PCM: $\mathbf{w}^{EM}$ is efficient

$$\mathbf{A}_\delta = \begin{pmatrix} 1 & \delta x_1 & x_2 & \dots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \dots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \dots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \dots & 1 \end{pmatrix} \in \mathcal{PCM}_n.$$

Theorem (Ábele-Nagy, Bozóki, 2016):
The principal right eigenvector of a simple perturbed pairwise comparison matrix is efficient.

Proof is based on the explicit formulas of $\mathbf{w}^{EM}$.

Double perturbed PCM ($n \geq 4$)

$$\begin{pmatrix} 1 & \delta x_1 & \gamma x_2 & x_3 & \cdots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \frac{x_3}{x_1} & \cdots & \frac{x_{n-1}}{x_1} \\ \frac{1}{\gamma x_2} & \frac{x_1}{x_2} & 1 & \frac{x_3}{x_2} & \cdots & \frac{x_{n-1}}{x_2} \\ \frac{1}{x_3} & \frac{x_1}{x_3} & \frac{x_2}{x_3} & 1 & \cdots & \frac{x_{n-1}}{x_3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \frac{x_3}{x_{n-1}} & \cdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \delta x_1 & x_2 & x_3 & \cdots & x_{n-1} \\ \frac{1}{\delta x_1} & 1 & \frac{x_2}{x_1} & \frac{x_3}{x_1} & \cdots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \gamma \frac{x_3}{x_2} & \cdots & \frac{x_{n-1}}{x_2} \\ \frac{1}{x_3} & \frac{x_1}{x_3} & \frac{x_2}{\gamma x_3} & 1 & \cdots & \frac{x_{n-1}}{x_3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \frac{x_3}{x_{n-1}} & \cdots & 1 \end{pmatrix}$$

Double perturbed PCM: $\mathbf{w}^{EM}$ is efficient

Theorem (Ábele-Nagy, Bozóki, Rebák, 2016):
The principal right eigenvector of a double perturbed pairwise comparison matrix is efficient.

Proof is based on the explicit formulas of $\mathbf{w}^{EM}$ and the characterization of efficiency by a strongly connected digraph.

$$\mathbf{A}(p,q) = \begin{pmatrix} 1 & p & p & p & \dots & p & p \\ 1/p & 1 & q & 1 & \dots & 1 & 1/q \\ 1/p & 1/q & 1 & q & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 1/p & 1 & 1 & 1 & \dots & 1 & q \\ 1/p & q & 1 & 1 & \dots & 1/q & 1 \end{pmatrix},$$

Proposition. (Bozóki, 2014):
Let q be positive and $q \neq 1$. Then $\mathbf{w}^{EM}$ is internally inefficient, therefore inefficient. Furthermore, CR inconsistency can be arbitrarily small if q is close enough to 1.

Weak efficiency

Definition: weight vector $\mathbf{w}$ is called *weakly efficient,* if there exists no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \dots, w'_n)^\top$ such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| < \left| a_{ij} - \frac{w_i}{w_j} \right| \qquad \text{for all } 1 \leq i \neq j \leq n.$$

Theorem (Bozóki, Fülöp, 2016):
The principal eigenvector of a pairwise comparison matrix is weakly efficient.

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Thank you for attention.

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