Recursive Estimation of ARX Systems Using Binary Sensors with Adjustable Thresholds

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Outline

- Problem: identifying an ARX systems via binary sensors
- Previous solutions typically assumed fully known noise characteristics
- They also assumed that the input signal can be chosen by the user
- We try to reduce the assumptions on the noise and the input
- Full knowledge on the distribution is not needed; the input is only observed
- But, the threshold of the binary sensor can be controlled $\sim$ dither signal
- Here, two recursive identification algorithms are proposed
- Algorithm I: FIR approximation; it is proved to be strongly consistent
- Algorithm II: simultaneous state and parameter estimation (simulations)

Structural Overview

PART I. **Problem Setting**
(ARX System via Binary Sensors, Dithering, Assumptions)

PART II. **General Form of the Algorithms**
(Sign-Error, Step-Sizes, Expanding Truncation Bounds)

PART III. **Recursive Identification: Algorithms I and II**
(FIR Approximation, Strong Consistency, Simultaneous Estimation)

PART IV. **Experimental Results**
(Simulation: Algorithms I and II on an ARX(2,2) System)

PART V. **Summary and Concluding Remarks**
(Main Ideas, Contributions and Highlights)

Problem Setting

- We observe an ARX system via a binary sensor:

$$X_t \triangleq \sum_{i=1}^{p} a_i^* X_{t-i} + \sum_{i=1}^{q} b_i^* U_{t-i} + N_t,$$
$$Y_t \triangleq \mathbb{I}(X_t \leq C_t),$$

where X_t — output (hidden state), U_t — input, N_t — noise (at time t)

- The thresholds of the binary sensor, $(C_t)_t$, can be controlled at each t
- Data: the inputs $(U_t)_t$ and the binary outputs $(Y_t)_t$ are observed
- Aim: to identify (estimate) $\theta^* = (a_1^*, \dots, a_p^*, b_1^*, \dots, b_q^*) \in \mathbb{R}^{p+q}$

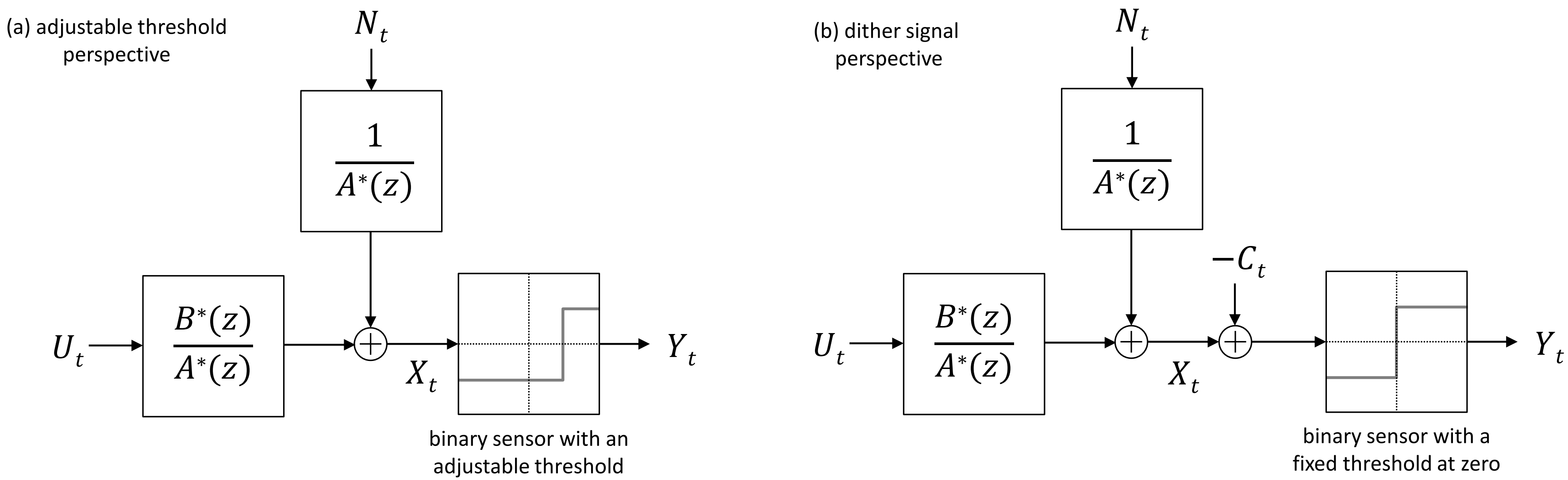
Adjustable Thresholds $\sim$ Dithering

- The binary output can be rewritten as

$$Y_t = \mathbb{I}(\varphi_t^{\mathrm{T}}\theta^* + N_t \leq C_t) = \mathbb{I}(\varphi_t^{\mathrm{T}}\theta^* + N_t - C_t \leq 0),$$

where $\varphi_t = (X_{t-1}, \dots, X_{t-p}, U_{t-1}, \dots, U_{t-q})$ — random regressor

- Choosing the threshold is equivalent to dithering



System Assumptions

- $(N_t)_t$ is i.i.d., continuous, zero mean, zero median, has a finite variance: $\sigma_n^2 \triangleq \mathbb{E}\left[\, N_t^2 \,\right] < \infty$, and has a continuous and positive density at zero
- $(U_t)_t$ is i.i.d., zero mean, $(U_t)_t$ and $(N_t)_t$ are independent, and $0 < \sigma_u^2 < \infty$, where $\sigma_u^2 \triangleq \mathbb{E}\left[\, U_t^2 \,\right]$
- The system is stable, i.e., the roots of $A^*(z)$ lie strictly inside the unit circle; additionally, the transfer function $B^*(z)/A^*(z)$ is irreducible,

$$\begin{aligned} A^*(z) &\triangleq 1 - a_1^* z^{-1} - a_2^* z^{-2} - \cdots - a_p^* z^{-p}, \\ B^*(z) &\triangleq b_1^* z^{-1} + b_2^* z^{-2} + \cdots + b_q^* z^{-q}, \end{aligned}$$

  where z^{-1} is the backward shift operator, $z^{-i} x_t \triangleq x_{t-i}$.
- The orders p and q are known

General Form of the Algorithms

- The general form of both proposed algorithms is

$$\hat{\theta}_{t+1} \;=\; \Pi_{M_{\mu(t)}}\Big[\, \hat{\theta}_t + \alpha_t\, \widehat{\varphi}_t \Big(1 - 2\, \mathbb{I}(X_t \leq \widehat{\varphi}_t^{\mathrm{T}} \hat{\theta}_t) \Big) \,\Big],$$

  where $\widehat{\varphi}_t$ is a regression vector defined differently in the two algorithms, $(\alpha_t)_t$ is a sequence of step-sizes and $\Pi_{M_{\mu(t)}}$ is a sequence of projections

- Assuming that N_t is continuous, we ($\mathbb{P}$-a.s.) have

$$\text{sign}(X_t - \widehat{\varphi}_t^{\mathrm{T}} \hat{\theta}_t) \;=\; 1 - 2\, \mathbb{I}(X_t \leq \widehat{\varphi}_t^{\mathrm{T}} \hat{\theta}_t),$$

- Thus, the above algorithm will behave almost surely as

$$\hat{\theta}_{t+1} \;=\; \Pi_{M_{\mu(t)}}\Big[\, \hat{\theta}_t + \alpha_t\, \widehat{\varphi}_t\, \text{sign}(X_t - \widehat{\varphi}_t^{\mathrm{T}} \hat{\theta}_t) \,\Big],$$

  which is a sign-error type algorithm with expanding truncation bounds

Step-Sizes

- Typical step-size assumption of stochastic approximation algorithms

$$\sum_{t=0}^{\infty} \alpha_t = \infty,$$

$$\sum_{t=0}^{\infty} \alpha_t^2 < \infty,$$

$$\forall\, t \geq 0 : \alpha_t \geq 0.$$

  The second condition can often be weakened to $\lim_{t\to\infty} \alpha_t = 0$

- Here, we will simply assume that

$$\alpha_0 = 1 \qquad \text{and} \qquad \forall\, t > 0 : \alpha_t = 1/t.$$

Expanding Truncation Bounds

- Let $(M_t)_t$ be a sequence of (strictly) monotone increasing positive real numbers with $M_t \to \infty$ as $t \to \infty$,
- Let $\mathbb{I}(\cdot)$ be the indicator function and define $\mu(t)$ and $\Delta\hat{\theta}_i$ as

$$\mu(t) \triangleq \sum_{i=1}^{t-1} \mathbb{I}\big(|\hat{\theta}_i + \Delta\hat{\theta}_i| > M_{\mu(i)}\big),$$

$$\Delta\hat{\theta}_i \triangleq \alpha_i\, \widehat{\varphi}_i \big(1 - 2\,\mathbb{I}(X_i \leq \widehat{\varphi}_i^{\mathrm{T}} \hat{\theta}_i)\big).$$

- Given a positive real M, projection Π_M is

$$\Pi_M(x) \triangleq \begin{cases} x & \text{if } \|x\| \leq M, \\ 0 & \text{otherwise.} \end{cases}$$

Algorithm I: FIR Approximation

- Using impulse responses, $(c_i^*)_{i=1}^{\infty}$ and $(d_i^*)_{i=0}^{\infty}$, we have

$$X_t \;=\; \sum_{i=1}^{\infty} c_i^* U_{t-1} + \sum_{i=0}^{\infty} d_i^* N_{t-i},$$

- Let's approximate our ARX system with an FIR system of order $p+q$

$$\boxed{X_t \;=\; \bar{\varphi}_t^{\mathrm{T}} \bar{\theta}^* + W_t,}$$

$$\bar{\varphi}_t \;\triangleq\; (U_{t-1}, \ldots, U_{t-p-q})^{\mathrm{T}}, \qquad \bar{\theta}^* \;\triangleq\; (c_1^*, \ldots, c_{p+q}^*)^{\mathrm{T}}.$$

- W_t is simply the unmodelled part of the system

$$W_t \;\triangleq\; \sum_{i=p+q+1}^{\infty} c_i^* U_{t-i} + \sum_{i=0}^{\infty} d_i^* N_{t-i}.$$

Algorithm I: FIR Approximation

- If we can estimate $\bar{\theta}^*$, we can also estimate the true parameter vector θ^*
- There is a function f, which we use for post processing, such that

$$\theta^* \; = \; f(\bar{\theta}^*),$$

- Algorithm I is defined by using $\widehat{\varphi}_t \triangleq \bar{\varphi}_t$ in the General Algorithm

Theorem 1 (Strong Consistency of Algorithm I)**.** *Let $(\hat{\theta}_t)_{t=0}^{\infty}$ be the sequence generated by Algorithm I (i.e. $\widehat{\varphi}_t = \bar{\varphi}_t$). Then, under the given assumptions, $f(\hat{\theta}_t)$ converges $(\mathbb{P}$-a.s.$)$ to θ^*, as $t \to \infty$, for any $\hat{\theta}_0 \in \mathbb{R}^{p+q}$.*

- Furthermore, $\sqrt{t}(\hat{\theta}_t - \bar{\theta}^*)$ is approximately normal

Algorithm II: Simultaneous Estimation

- Main idea: to achieve a direct estimate of θ^* by simultaneously maintaining an estimate for the output, $\widehat{X}_t$ and for the parameter, $\hat{\theta}_t$, at time t.
- The sequence of output estimates is defined as

$$\widehat{X}_t \triangleq \begin{cases} \sum_{i=1}^{p} \hat{a}_{t,i} \widehat{X}_{t-1} + \sum_{i=1}^{q} \hat{b}_{t,i} U_{t-i} & \text{if } t \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $(\hat{a}_{t,i})_{i=1}^{p}$ and $(\hat{b}_{t,i})_{i=1}^{q}$ are the estimates of the true parameters.

- Algorith II: is defined by setting the General Algorithm as

$$\begin{aligned} \widehat{\varphi}_t &\triangleq (\widehat{X}_{t-1}, \dots, \widehat{X}_{t-p}, U_{t-1}, \dots, U_{t-q})^{\mathrm{T}}, \\ \hat{\theta}_t &\triangleq (\hat{a}_{t,1}, \dots, \hat{a}_{t,p}, \hat{b}_{t,1}, \dots, \hat{b}_{t,q})^{\mathrm{T}}. \end{aligned}$$

Simulation Experiment: ARX(2, 2)

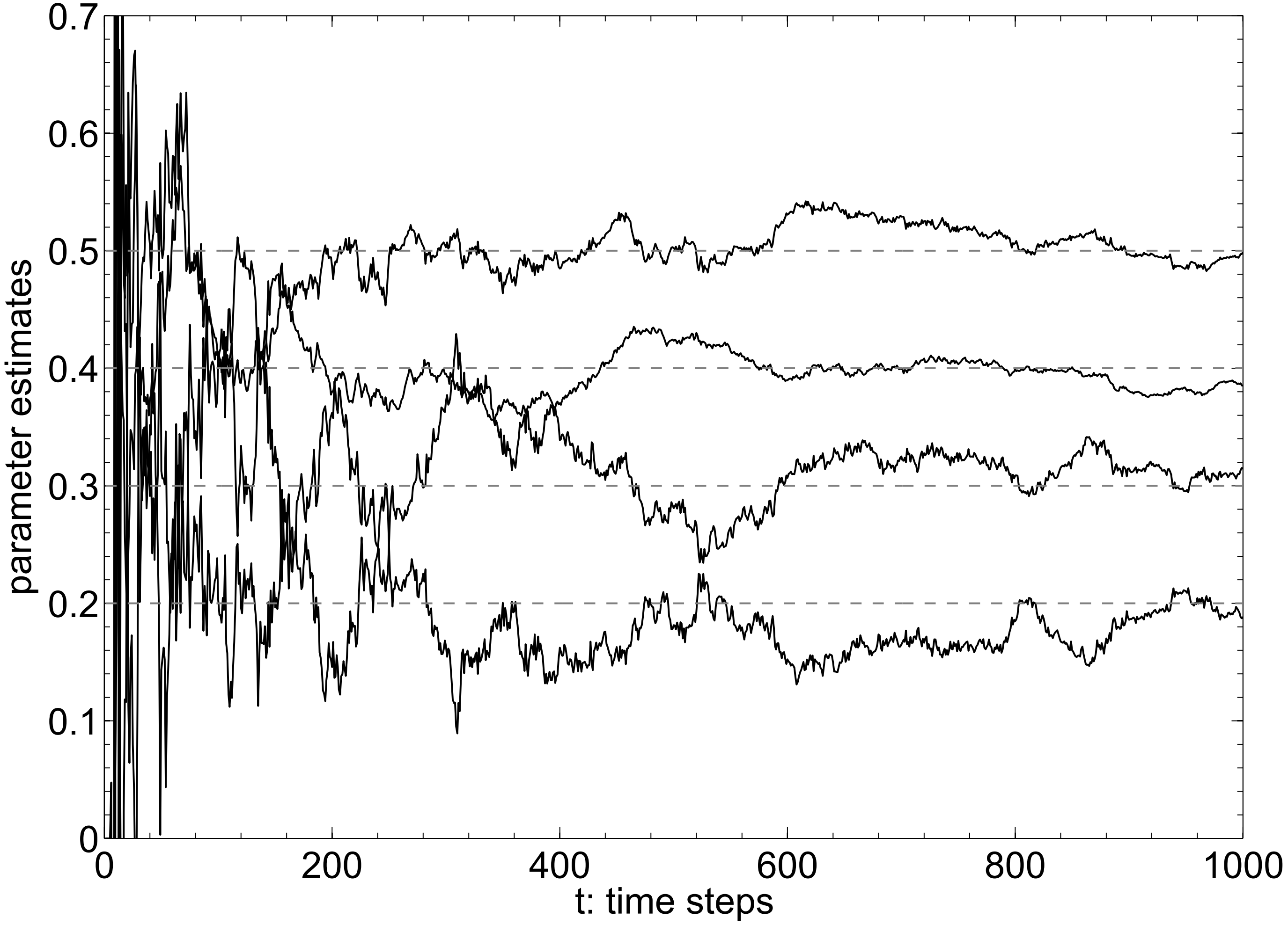


Figure 1: Recursive estimation with Algorithm I

Simulation Experiment: ARX(2, 2)

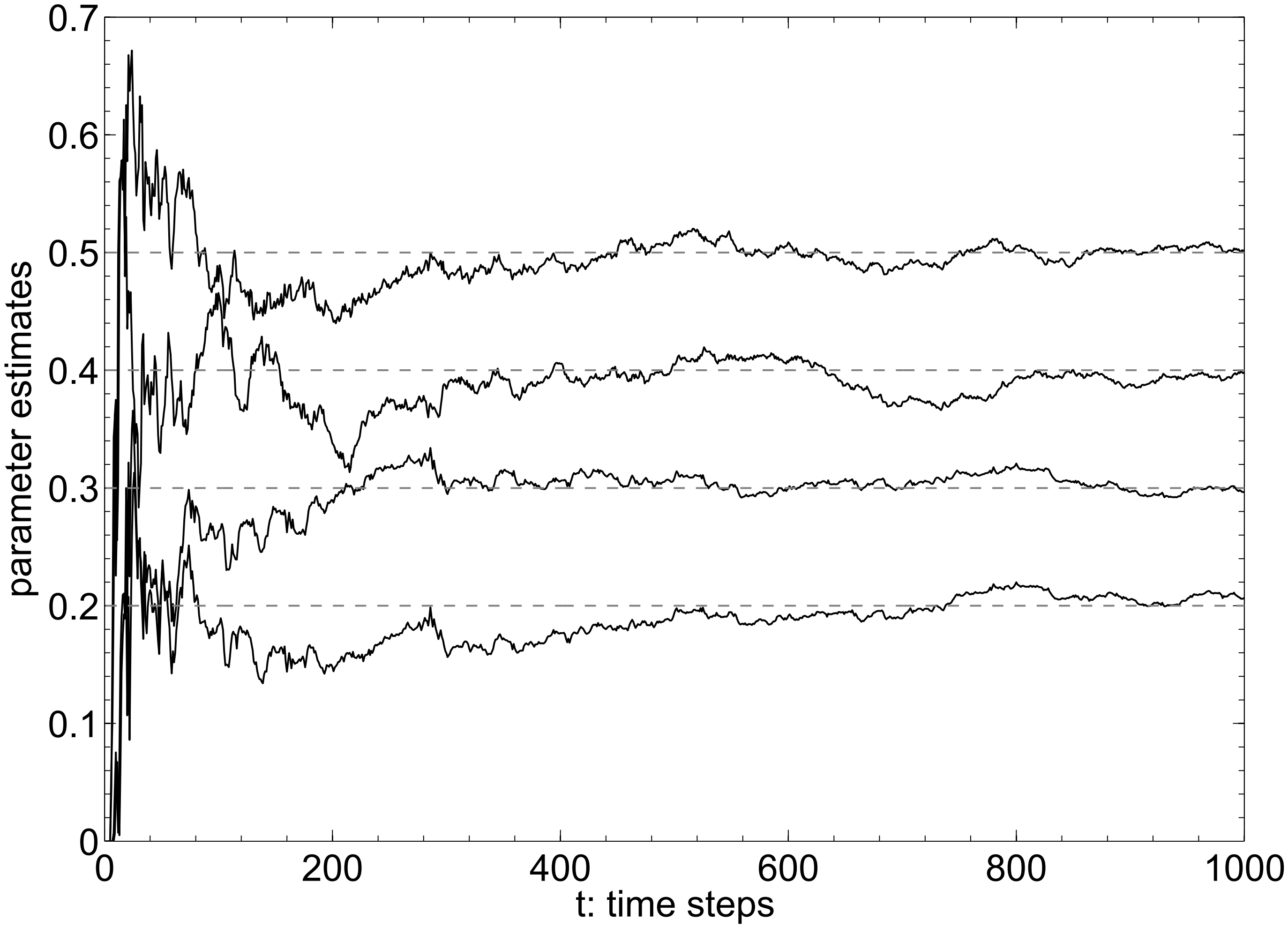


Figure 2: Recursive estimation with Algorithm II

Summary and Concluding Remarks

- Two recursive identification algorithms have been proposed for identifying ARX systems via binary sensors
- These algorithms neither assume the knowledge of the particular noise distributions, nor assume that the input signal can be chosen by the user
- But, they do assume that the threshold of the sensor can be controlled
- This is assumption is equivalent to allowing a dither signal
- Algorithm I: FIR approximation; it was proved to be strongly consistent
- Algorithm II: simultaneous state and parameter estimation (no theorem)
- Experimental results demonstrated that both algorithms efficiently approximated the parameters of an ARX(2,2) system

Thank you for your attention!

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