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IN DEFENSE OF THE SYMMETRY OF TRUE AND FALSE

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Abstract: *According to the anti-realist views, mathematics is the creation of the mind and mathematical statements do not have any truth-value independently of our knowledge. In this paper I give a brief overview on anti-realist mathematical theories, such as constructivism and intuitionism and, by arguing that Husserl's criticism of psychologism can be applied to attack these anti-realist concepts, I try to defend the classical mathematical realist (or Platonist) view of eternal truths.*

1 ANTI-REALISM IN MATHEMATICS

Most mathematicians accept a *realist* (also called as *Platonist*) view on mathematical statements. Realism according to (Dummett, 1982) is a *semantic* thesis which asserts that statements in a given class relate to some reality that exists independently of our knowledge of it, in such way, that reality renders each statement in the class determinately true or false, again independently of whether we know, or even able to discover its truth-value. Note that Gödel's incompleteness theorems (1931) do not attack this approach, they only talk about *provability*, and moreover, Kurt Gödel himself was realist (Gödel, 1951). Mathematical realism was almost universally accepted until the 19th century, when *anti-realist* mathematical theories started to appear. These theories seem to accept the Protagorean formula that "*man is the measure of all things*". According to them, mathematics is determined by our minds, mathematical objects and statements are just our own creations and there does not exists any "transcendental" reality which makes these statements true or false independently of our reasoning. These theories are called *constructivism* or *intuitionism*. One of the main characteristics of them is the strict interpretation of the phrase "there exists" as "we can construct". They, however, are not homogeneous; there are considerable differences between the various representatives, for a detailed overview, see (Troelstra and Van Dalen, 1988).

Intuitionist mathematics can be traced back to Immanuel Kant, who in his "Critique of Pure Reason" treated mathematical statements as *synthetic a priori* (Kant, 1787), and not as *analytic* truths, which was the accepted view of that time (e.g., G. W. Leibniz). Kant's views can be clearly recognized, for example, in Brouwer's "intuition of time".

Probably the first mathematician who can be treated as constructivist was Kronecker, who in the 19th century started an arithmetization program, in which he wanted to "arithmetize" Algebra and Analysis. One consequence of his efforts was that he considered a definition acceptable, if it could be checked in a finite way. This viewpoint led him to the criticism of "pure" existence proofs. A remark of him captures well his ideas: "*the Lord made the natural numbers, everything else is the work of men*". Later, Kronecker's work was continued by Julies Molk. After Kronecker, the French semi-intuitionists, such as Baire, Borel, Lebesgue, Lusin, Poincaré, expressed more or less constructivist views when they attacked the axiom of choice, the well-ordering of the continuum, the Cantorian set-theory and the mathematical logic. Their critique on logic can be illustrated by Poincaré's words: "*The syllogism can teach us nothing essentially new*". Borel and Lebesgue argued that only the effectively (i.e., by finitely many words) defined objects exist in mathematics, and consistency is not sufficient for existence, however, Borel treated the continuum as independently given by our intuition.

The first foundations of a precise, systematic constructive mathematics were given by the Dutch mathematician L. E. J. Brouwer. According to him, mathematics is a free creation of the mind, mathematical objects are mental constructs, and mathematics is independent of any language or Platonic reality. Therefore, there do not exist mathematical truths independently of our knowledge. These views led him to constructive mathematics, in which a large part of classical mathematics is rejected. The consequences of these views are: mathematics is independent of logic, logic is just applied mathematics and mathematics cannot be founded upon axiomatic methods. Brouwer has showed that with his views on mathematics, one cannot hold to the *principle of the excluded middle* (PEM). He has constructed several "counterexamples" to refute PEM. Although Brouwer rejected formalism, his pupil Heyting has developed formal intuitionistic systems, such as intuitionistic propositional and predicate logic and arithmetic (which is the Peano arithmetic with intuitionistic logic). Later Gentzen and Kleene extended his results on intuitionistic mathematics; Glivenko, Gentzen and Gödel have proved (independently) that the classical and intuitionistic (propositional and predicate) logic is equiconsistent. Semantics to intuitionistic logic was given by Kleene, Beth, Aczél and Kripke (Moschovakis, 2004). Later Markov further developed constructive mathematics on the basis of recursive functions. There was a more radical version of constructivism, called *finitism*, which criticized the use of abstract notions. For example, Skolem and Goodstein proposed a very narrow version of constructivism, in which only concrete combinatorial operations on strictly finite mathematical objects are allowed (such as a table of multiplication and a natural number).

Many mathematicians believed, that the non-classical principles (especially because of the omission of PEM) are not strong enough for modern mathematics, e.g., Hilbert wrote: "*Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists.*" (1928). These views have changed radically, when Bishop published his book in which he rebuilt a large part of 20th century analysis on the basis of intuitionistic logic (Bishop, 1967). He gave the intuitionistic counterparts of theorems, such as the Stone-Weierstrass Theorem, the Hahn-Banach and separation theorems, the spectral theorem for self-adjoint operators on a Hilbert space, the Lebesgue convergence theorems for abstract integrals, Haar measure and the abstract Fourier transform, etc. Therefore, even without PEM, constructivism can be a rival of classical mathematics.

The philosophical viewpoint of intuitionism was defended by several authors. For example (Dummett, 1973) and (Prawitz, 1977) argued with the Wittgensteinian "meaning as use" theory. Wittgenstein wrote that: "*For a large class of cases – though not for all – in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language*" (Wittgenstein, 1958). Applying this to mathematics, the meanings of mathematical statements are given by their proofs and the way they behave in proofs of other statements. According to this, they argue that there is a problem with the meaning of undecidable statements, which seems to lack meaning. Pourciau states that the acceptances of some basic principles (which almost seem self-evident) involve intuitionism (Pourciau, 1999). These principles are: (M) know what something means before you ask if it is true, (A) build in no clearly unwarranted assumptions, (S) move from the simple to the less simple. In (Pourciau, 2000) he argues that Khunian revolutions in mathematics are logically possible and intuitionism had the chance of a scientific revolution, but due to "accidental historical factors" it had failed.

2 IN DEFENSE OF MATHEMATICAL REALISM

In this section I try to very briefly recall Husserl's criticism on the psychological foundations of logic. I think that Husserl's argument directly applies to anti-realist mathematics. In the first part of his Logical Investigations Husserl criticized psychology as a foundation of logic, because on the basis of it, one cannot achieve true knowledge (Husserl, 1900). He thought that the combination of psychology and logic can only lead to skepticism, because psychology cannot ground the absolute necessity of logical laws. That is why he introduced "pure logic" which he later revised and renamed to “transcendental phenomenology”. Psychologism as a basis of logic leads to skepticism, which means there is “*no truth, no knowledge, no justification of knowledge*”. Another meaning of skepticism is the “*limit of knowledge to mental existence, and would deny the existence or knowability of things in themselves*”. According the psychologism, the truth is “*relative to the contingently judging subject*” (Husserl, 1900). One can argue against psychological skeptical relativism by way that the very formulation of this doctrine denies what is subjectively or objectively a condition of its own validity. It asserts from itself that it is a universal truth, however it also states, that there are not any universal truths, everything is relative to the mind. It is a clear contradiction.

In my opinion even Gödel’s incompleteness theorems can be used to defend realism. Constructivism states that a mathematical statement is true if and only if somebody has constructed a proof of it in his mind; however, the incompleteness theorems assert that truth cannot be equated with provability in any effectively axiomatizable theory.

References

Bishop, E. (1967) *Foundations of Constructive Analysis*, New York: McGraw-Hill

Dummett, M. (1982) Realism, *Synthese*, 52, 55–112

Dummett, M. (1973) The Philosophical Basis of Intuitionistic Logic, Reprinted in: *Truth and Other Enigmas*, Duckworth, 1978, 215–247

Gödel, K. (1951) Some Basic Theorems on the Foundations of Mathematics and their Implications (Gibbs Lecture) In: *Kurt Gödel: Collected Works*, Oxford University Press, 1995, 304–323

Husserl, E. (1900) *Logische Untersuchungen, Erster Teil: Prolegomena zur reinen Logik* (English trans. by J. N. Findlay, English title: Logical Investigations, London: Routledge, 1900/01; 1913)

Kant, I. (1787) *Kritik der reinen Vernunft*, In: Immanuel Kant: Gesammelte Schriften, Herausgegeben von der Königlich Preußischen Akademie der Wissenschaften, Erste Abteilung, 3, Berlin, 1911 (Hungarian trans. by Kis J. In: A tiszta ész kritikája, Budapest: Ictus Kiadó, 1995)

Moschovakis, J. (2004) Intuitionistic Logic, In: Zalta, E. N. ed.: *The Stanford Encyclopedia of Philosophy*, 2004; http://plato.stanford.edu/archives/spr2004/entries/logic-intuitionistic/

Pourciau, B. (2000) Intuitionism as a (Failed) Kuhnian Revolution in Mathematics, *Studies in History and Philosophy of Science*, 31, no. 2, 297–329

Pourciau, B. (1999) The Education of a Pure Mathematician, *The American Mathematical Monthly*, October 1999, 106, no. 8, 720–732

Prawitz, D. (1977) Meaning and Proofs: On the Conflict Between Classical and Intuitionistic Logic, *Theoria*, 43, 2-40 (Hungarian trans. by Csaba F., In: Csaba F. ed.: A matematika filozófiája a 21. század küszöbén, Budapest: Osiris Kiadó, 2003, 123–163)

Troelstra, A. S. and Van Dalen, D. (1988) *Constructivism in Mathematics*, 1, Amsterdam: North-Holland

Wittgenstein, L. (1958) *Philosophical Investigations*, Oxford: Basil Blackwell (Hungarian trans. by Neumer K., In: Filozófiai vizsgálódások, Budapest: Atlantisz Kiadó, 1998)