Finding equilibrium of bimatrix games

The Lemke - Howson algorithm

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Importance of Nash equilibrium in game theory

> *"Together with factoring, the complexity of finding a Nash equilibrium is in my opinion the most important concrete open question on the boundary of P today."*

C. H. Papadimitriou

It is unknown, whether an equilibrium point can be found in polynomial time, even in 2-player games. Also the existence of polynomial-time approximation schemes are not known.

Brief history

2-player games

- 1964 Lemke - Howson
  special case of the Linear Complementarity Problem (LCP)
- 2004 Porter - Nudelman - Shoham
  AI methods (namely Constraint Programming)
- 2005 Sandholm - Gilpin - Conitzer
  Mixed Integer Programming (MIP) problem with Branch&Cut algorithm and various heuristics

n-player games

- 1987 van der Laan - Talman - van der Heyden
  Simplical Subdivision
- 2003 Govindan - Wilson
- 2004 Porter - Nudelman - Shoham
  AI methods (CP)

Bimatrix games

- Two-person, non-zero-sum games
- Payoff matrices: $A, B \in \mathbb{R}^{m \times n}$
- Mixed strategies: $x \in \Delta_m = \{\, x \in \mathbb{R}^m : x \geq 0 \wedge x^T e = 1 \,\}$, $y \in \Delta_n$
- Payoffs: $x^T Ay$ and $x^T By$
- Equilibrium point:
  $(x_0, y_0) \in \Delta_m \times \Delta_n \Leftrightarrow \forall x \in \Delta_m, \forall y \in \Delta_n :$

$$x_0^T Ay_0 \geq x^T Ay_0 \text{ and } x_0^T By_0 \geq x_0^T By$$

An equivalent formulation #1

Lemma. $(x_0, y_0) \in \Delta_m \times \Delta_n$ is an equilibrium point $\Leftrightarrow$

$$(x_0^T A y_0) e \geq A y_0 \text{ and } (x_0^T B y_0) e \geq B^T x_0$$

Proof.

$\Rightarrow$: $\forall i \in \{1, \ldots, m\} : e_i \in \Delta_m \Rightarrow x_0^T A y_0 \geq e_i^T A y_0 = (A y_0)_i$

$$\Rightarrow (x_0^T A y_0) e \geq A y_0$$

$\Leftarrow$: $\forall x \in \Delta_m : x^T (x_0^T A y_0) e \geq x^T A y_0$ but

$$x^T (x_0^T A y_0) e = x_0^T A y_0 \text{ since } x^T e = 1$$

QED

An equivalent formulation #2

Theorem. $(x_0, y_0) \in \Delta_m \times \Delta_n$ is an equilibrium point $\Leftrightarrow$ for a fixed $k \in \mathbb{R}$ such that $kE - B^T > 0$ and $kE - A > 0$, where E is the matrix with all 1's, $x := \frac{x_0}{k - x_0^T B y_0}$ and $y := \frac{y_0}{k - x_0^T A y_0}$ are solutions of the following problem:

$$
\begin{aligned}
&(kE - B^T)x \geq e \\
&(kE - A)y \geq e \\
&y^T\left((kE - B^T)x - e\right) = 0 \\
&x^T\left((kE - A)y - e\right) = 0 \\
&x, y \geq 0
\end{aligned}
$$

Proof. Note that $Ex_0 = Ey_0 = e$ and $x_0^T E y_0 = 1$.
From the lemma, we have: $ke - B^T x_0 \geq ke - (x_0^T B y_0)e$.
$ke - B^T x_0 = kEx_0 - B^T x_0 = (kE - B^T)x_0$ and
$ke - (x_0^T B y_0)e = (k - x_0^T B y_0)e$ which yields
$(kE - B^T)x_0 \geq (k - x_0^T B y_0)e$ and since $kE - B^T > 0 \Rightarrow k > x_0^T B y_0$ we have
$(kE - B^T)x \geq e$
$y^T\left((kE - B^T)x - e\right) = 0$ and $x \geq 0$ are trivial.

Let $x_0 := (\frac{1}{x^T e})x$ and $y_0 := (\frac{1}{y^T e})y$. Thus $x_0^T A y_0 = k - (\frac{1}{y^T e})$ and
$x_0^T B y_0 = k - (\frac{1}{x^T e})$.
Then it is easy to prove, that (x_0, y_0) is equilibrium point. **QED**

The final form

From now on we consider the following problem:

$$
\begin{aligned}
&B^T x - e \geq 0 \\
&Ay - e \geq 0 \\
&y^T \left(B^T x - e\right) = 0 \\
&x^T \left(Ay - e\right) = 0 \\
&x, y \geq 0
\end{aligned}
$$

and the solution (x, y) will be called equilibrium point.

Geometric interpretation

$$X := \{ x \in \mathbb{R} : x \geq 0 \wedge B^T x - e \geq 0 \}$$

$(B, I) \in \mathbb{R}^{m \times (n+m)}$

$x \in X \Leftrightarrow$

i.) $e_i^T x \geq 0$ $\qquad i \in \{1, \dots, m\}$

ii.) $b_j^T x - 1 \geq 0$ $\qquad j \in \{1, \dots, n\}$

$\forall x$ let $M(x)$ be the matrix that consists b_j and e_i columns from $(B, I) \Leftrightarrow b_j^T x - 1 = 0$ and $e_i^T x = 0$. (It is possible, that $M(x)$ has no columns!)

$x \in X$ is an extreme point of $X \Leftrightarrow \operatorname{rank}(M(x)) = m$.

Nondegeneracy assumption:
$\forall x : M(x) \in \mathbb{R}^{m \times r} \Rightarrow \operatorname{rank}(M(x)) = r$.

Without proof: w.l.o.g. nondegeneracy can be assumed

The effect of nondegeneracy

For $x_0 \in X$ let $M(x_0) = (d_1, \dots, d_r)$
Construct a non-singular $C := (d_1, \dots, d_r, \dots, d_m)$ and let $(C^T)^{-1} = (d^1, \dots, d^m)$. Then $d_i^T d^j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Lemma. $\exists k > 0, \forall t_1, \dots, t_m \in \mathbb{R} : \sum_{i=1}^{m} t_i^2 \leq k \wedge t_1, \dots, t_r \geq 0 \Rightarrow$
$x = x_0 + \sum_{i=1}^{m} t_i d^i \in X$.

Proof. Let d a column of (B, I), and $d^T x = d^T x_0 + \sum_{i=1}^{m} t_i (d^T d^i)$

If $d = d_s$ $(1 \leq s \leq r) \Rightarrow d^T x = \theta + t_s \geq \theta$, where
$\theta = \begin{cases} 1, & d \text{ is from } B \\ 0, & d \text{ is from } I \end{cases}$
If $d = d_s$ $(r < s \leq m) \Rightarrow d^T x_0 > \theta \Rightarrow d^T x > \theta$ for sufficiently small t_s.
$\Rightarrow x \in X$ **QED**

Corollaries

i.) If x_0 is an extreme point of $X \Rightarrow \forall t_i \geq 0$ "small enough": $x = x_0 + t_i d^i \in X$.
If $t_i > 0 \Rightarrow M(x) = M_i$, which is $M(x_0)$ without the column d_i.
The set of such points is an open edge of X with endpoint x_0.

ii.) If $x_0 \in X$, $\operatorname{rank}(M(x_0)) = m - 1 \Rightarrow \exists k > 0, \forall t_m : |t_m| \leq k \Rightarrow x = x_0 + t_m d^m \in X$ and $M(x) = M(x_0)$.
The set of such points is an open edge of X.

iii.) If x_0 is an extreme point of X, there exist precisely m open edges of X with endpoint x_0.

iv.) There are precisely m unbounded edges of X, and each has one endpoint. These points are in the form $x = ke_i$ with $k > 0$ and "large enough" (since $B > 0$).
Any other edges have two endpoints, called adjacent extreme points whose M matrices differ only in one column.

Analogous geometric interpretation

$$Y := \{\, y \in \mathbb{R} : y \geq 0 \wedge Ay - e \geq 0 \,\}$$

$$\left(I, A^T\right) \in \mathbb{R}^{n \times (n+m)}$$

$y \in Y \Leftrightarrow$

i.) $a_i^T y - 1 \geq 0$ $\qquad i \in \{1, \ldots, m\}$

ii.) $e_j^T y \geq 0$ $\qquad j \in \{1, \ldots, n\}$

$N(y)$ matrix.

Nondegeneracy assumption.

Cartesian product

$Z := X \times Y$

Thus $z = (x, y) \in Z$ is equilibrium point $\Leftrightarrow$

i.) $(e_i^T x)(a_i^T y - 1) = 0 \qquad i \in \{1, \dots, m\}$

ii.) $(e_j^T y)(b_j^T x - 1) = 0 \qquad j \in \{1, \dots, n\}$

(equilibrium conditions).

$z = (x, y)$ is an extreme point of $Z \Leftrightarrow x$ is an extreme point of X and y is an extreme point of Y.

z lies on an open edge of $Z \Leftrightarrow x$ or y is an extreme point, the other is not, but lies on an open edge.

Lemma #1

Lemma. $z = (x, y) \in Z$ is equilibrium point $\Rightarrow$ z is an extreme point of Z.

Proof. If z is equilibrium, then the equilibrium conditions must hold, and therefore $(M(x), N(y))$ matrix must have at least $n + m$ columns. But (from the nondegeneracy) $M(x)$ can have at most m, and $N(y)$ can have at most n columns, thus z must be an extreme point. **QED**

Corollary. $z = (x, y) \in Z$ is equilibrium point
$\Rightarrow \forall r \in \{1, \dots, n + m\}$ only one of the followings holds:

i.) the rth column of (B, I) is a column of $M(x)$

ii.) the rth column of (I, A^T) is a column of $N(y)$

Lemma #2

$\forall r \in \{1, \ldots, n\}$ let $S_r \subseteq Z$ be the set of points, which satisfy the equilibrium conditions except maybe one, the $(e_r^T y)(b_r^T x - 1) = 0$.

Lemma. Each point of S_r is either an extreme point of Z, or a point on an open edge of Z.

Proof. $z \in S_r$ satisfies at least $n + m - 1$ equilibrium conditions, and therefore $(M(x), N(y))$ matrix must have at least $n + m - 1$ columns. But (from the nondegeneracy) $M(x)$ can have at most m, and $N(y)$ can have at most n columns, thus z is either an extreme point or lies on an open edge. **QED**

Lemma #3

Lemma. There is precisely one unbounded open edge of Z composed of point of S_r.

Proof. $k_0 := \min\{ k : ke_r \in Y \} \neq \emptyset$ and $y_0 := k_0 e_r$. Then y_0 is an extreme point of Y and $\exists! s \in \{1, \ldots, m\} : a_s^T y_0 = 1$.
$k_1 := \min\{ k : ke_s \in X \} \neq \emptyset$ and $x_0 := k_1 e_s$.
$\forall k \geq k_1 (ke_s, y_0)$ points form an unbounded edge of Z, let it be called E_0.
It can be shown, that a point on an edge of Z may not belong to both S_r and $S_{r'}$ if $r \neq r'$.
There exist exactly n unbounded edges in Y, so every S_r contains one of them. **QED**

Lemma #4

Lemma. Let $z \in S_r$ be an extreme point of Z. There are then one or two open edges of Z, consisting wholly points of S_r, which have z as endpoint. If there is only one open edge $\Leftrightarrow$ z is an equilibrium point.

Proof.

$(e_r^T y)(b_r^T x - 1) = 0$: Only one of the factors equals 0. If $e_r^T y = 0$, then only one of the $m + n$ edges – where $e_r^T y > 0$ – is in S_r. (The $b_r^T x > 1$ case is analogous.)

$(e_r^T y)(b_r^T x - 1) > 0$: $\exists! t : e_t^T y = 0 \wedge b_t^T x = 1$. Then the two edges, where one of these two equalities does not hold, are in S_r.

QED

Paths in Z

Two open edges of Z are adjacent if they have a common endpoint.

A sequence of adjacent open edges in S_r is called an r-path.

A cyclic r-path is said to be closed.

An acyclic r-path, which cannot be further extended, is said to be complete.

Note, that the number of extreme points is finite.

Theorem. S_r is the union of a finite number of disjoint r-paths. Each r-path is either closed (and thus contains no equilibrium point), or complete with one or two equilibrium points. Furthermore there exist exactly one complete r-path, P_0, which contains E_0, and thus only one equilibrium point.

Corollary. The number of equilibrium points is odd. One of them can be computed by traversing P_0 starting from the endpoint of E_0.

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