Classification Results for Non-Mixed and Mixed Optimal Covering Codes: a Survey

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***Abstract*— A survey is given that comprises the known classification results on non-mixed and mixed optimal covering codes. Several new, so far unpublished results of the author are declared and proved as well. The generality of the specified classification results are quite different. The proof of the more or less general new results are mostly combinatorial. In contrast with the latter, computer-aided proofs are given for numerous individual cases. The individual (old and new) classification results are merged and summarized in a set of tables.**

Keywords: covering codes, mixed codes, classification of codes.

I. Introduction

We consider covering codes in either non-mixed or mixed Hamming spaces denoted by $H = Z_q^n$ for non-mixed spaces, $H = Z_{q_1}^{n_1} Z_{q_2}^{n_2}$, $H = Z_{q_1}^{n_1} Z_{q_2}^{n_2} Z_{q_3}^{n_3}$ (etc.) for mixed spaces, where $Z_q = \{0, 1, \ldots, q-1\}$.

The Hamming distance $d(x, y)$ between two words $x, y \in H$ is the number of coordinates in which they differ. The covering radius of a code $C \subseteq H$ is the smallest positive integer R such that for an arbitrary $x \in H$, there exists one (or more) $y \in C$ with $d(x, y) \leq R$. For more details see Definition 1 in [8].

A covering code is called *optimal* if it has minimum cardinality among codes with given parameters n_i and R; the minimum cardinality is denoted by

$$K_q(n, R),\ K_{q_1,q_2}(n_1, n_2, R),\ K_{q_1,q_2,q_3}(n_1, n_2, n_3, R)$$

and analogous notations are used for arbitrary number of different q_is.

The exact values of $K.(\ldots)$ are known for only very few cases. For each exact value, naturally arises the question about classification. The minimum doing in a classification procedure is the determination of the number of *inequivalent* optimal covering codes for a given set of parameters, which can be followed by grouping them according some quantitative and/or qualitative properties. Two codes are said to be *equivalent* if there is a permutation of the coordinates and one permutation of the values for each of the coordinates that map one code into the other.

The research was supported in part by OTKA grants K 60480, K 77420.

The following invariants of codes can be studied in connection with classification: minimum distance, normality, surjectivity, balancedness, self-complementary etc. The definition of minimum distance and normality can be found in [8] as Definitions 2 and 3. Hereafter the definition of other essential properties will be given.

Definition 1: A code $C \subset H = Z_q^n$ is called balanced if for any coordinate $i \in \{1, 2, \ldots, n\}$ and for any symbol $j \in \{0, 1, \ldots, q-1\}$, the number of occurrences of the given symbol in the given coordinate position is either $\lfloor n/q \rfloor$ or $\lceil n/q \rceil$.

The extension of the notions of normality and balancedness for mixed codes is straightforward.

Definition 2: A binary code is called self-complementary if for any codeword c, the complementary word of c is also a codeword.

Definition 3: An s-surjective code is a code in $H = Z_q^n$ with the property that, in every s coordinate positions, all q^s possibilities occur at least once.

For a concise indication for normality or abnormality of the classified optimal covering codes, we introduce also the notation $L.(\ldots)$ for the smallest possible size of normal covering codes. Thus, let $L_q(n, R)$ denote the minimum number of codewords in any normal code $C \subset Z_q^n$ with covering radius at most R; $L_{q_1,q_2}(n_1, n_2, R)$, $L_{q_1,q_2,q_3}(n_1, n_2, n_3, R)$, etc. denote the minimum number of codewords in any mixed code $C \subset Z_{q_1}^{n_1} Z_{q_2}^{n_2}$, $Z_{q_1}^{n_1} Z_{q_2}^{n_2} Z_{q_3}^{n_3}$ etc. with covering radius at most R, which is normal for at least one coordinate of each type.

We have obvious classification result (uniqueness of the optimal covering code up to equivalence) in the two marginal cases when

- $R = 0$,
- $R = n$ for non-mixed, $R = \sum n_i$ for mixed codes.

In the first case, $K.(\ldots) = |H|$, i.e., the cardinality of the actual Hamming space, while in the second case, $K.(\ldots) = 1$.

In the rest of the paper it will be assumed that $R > 0$, and R is smaller than the number of coordinates.

The following two theorems are useful for some kinds of classification theorems regarding q-ary covering codes. The proof of Theorem 1 is routine, and therefore it is omitted, while Theorem 2 is known from [1].

Theorem 1: $K_q(n, R) = q$ if and only if $R < n < \frac{qR+q}{q-1}$. The optimal covering codes are exactly those q-word codes, for which there are at least $q(n - R - 1) + 1$ coordinate positions where all symbols of $0, 1, \ldots, q-1$ occur in the codewords.

Theorem 2: The number of integer solutions of system

$$\sum_{i=1}^{t}\sum_{j=1}^{k_i} n_{ij} = s$$

$$n_{i1} \geq n_{i2} \geq \ldots \geq n_{ik_i} \geq 0 \quad (i = 1, 2, \ldots, t)$$

is equal to the coefficient of s in the expansion of

$$\prod_{i=1}^{t}\prod_{j=1}^{k_i} \frac{1}{1 - x^j}.$$

It is easy to see as well, that all optimal codes described in Theorem 1 are normal with respect to the coordinates where all of the symbols appear, and consequently also $L_q(n, R) = q$ holds whenever $K_q(n, R) = q$.

II. Binary covering codes

For binary optimal covering codes with 2, 4 or 7 codewords, the following classification results are known, in general.

Theorem 3: ([6, Theorem 7.29]) $K_2(n, R) = 2$ if and only if $\lfloor \frac{n}{2} \rfloor \leq R < n$, and the number of inequivalent optimal covering codes is $2R - n + 2$.

Theorem 4: ([6, Theorem 7.30]) $K_2(n, R) = 4$ if and only if $n = 2R+2$, and the number of inequivalent optimal covering codes is $\lfloor (\frac{R}{2} + 1)^2 \rfloor$.

Theorem 5: ([11, Theorem 3.3]) $K_2(n, R) = 7$ if and only if $n = 2R+3$ where $R > 0$, and the number of inequivalent optimal covering codes is the coefficient of x^{R-1} in the expansion of

$$\frac{1}{(1-x)^3(1-x^2)^2(1-x^3)} =$$

$$= 1 + 3x + 8x^2 + 17x^3 + 33x^4 + 58x^5 + \ldots$$

The sequence of coefficients is A002625 in the on-line encyclopedia of integer sequences [21].

It is also known that all optimal covering codes belonging to $K_2(n, R) = 2$ are normal and those belonging to $K_2(n, R) = 4$ and $K_2(n, R) = 7$ are normal with respect to every coordinates. Consequently, $L_2(n, R) = K_2(n, R)$ always holds in these three special cases.

The remaining known classification results for binary covering codes (which are not covered by Theorems 3-5) are tabulated in Table I. These classification results are partly combinatorial (where the number of inequivalent codes, denoted by N, is not greater than 10), partly computational (where $N > 10$). It is evident from the table, that $L_2(n, R) = K_2(n, R)$ holds also for all of the tabulated binary classification results. A detailed analysis shows that all optimal covering codes belonging to the cases of Table I are normal in all coordinates, with the exception of $K_2(8, 2) = 12$ where only 265 from the 277 inequivalent optimal codes are normal in all coordinates, and $K_2(10, 3) = 12$ where only 9337 from the 11481 inequivalent optimal codes are normal in all coordinates. Furthermore, non-balanced codes among the optimal covering codes appear first at the same cases.

Example 1. Consider the two inequivalent optimal codes belonging to $K_2(6, 1) = 12$. These optimal codes have the following structures (in matrix arrangement):

$$\begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&1&1&1\\ 0&1&0&1&1&1\\ 1&0&0&1&1&1\\ 1&1&1&0&1&1\\ 1&1&1&1&0&1\\ 1&1&1&1&1&0\\ 0&1&1&0&0&0\\ 1&0&1&0&0&0\\ 1&1&0&0&0&0 \end{bmatrix} \qquad \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&1\\ 0&0&0&1&0&1\\ 0&0&1&1&1&0\\ 0&1&0&1&1&0\\ 1&0&0&1&1&0\\ 1&1&1&0&1&1\\ 1&1&1&1&0&1\\ 1&1&1&1&1&1\\ 0&1&1&0&0&0\\ 1&0&1&0&0&0\\ 1&1&0&0&0&0 \end{bmatrix}$$

It can be seen that the first optimal code has a highly symmetrical form. This code is self-complementary and 3-surjective. The second structure differs from the first one only in the last column, and it is neither self-complementary, nor 3-surjective, but it is balanced and 2-surjective. Their minimum distance is also different: 2 for the first, 1 for the second code.

Example 2. There are four inequivalent optimal codes for $K_2(9, 2) = 16$. All of them are balanced, having eight 0s and eight 1s in each coordinate position. Among the inequivalent structures two self-complementary codes and only one 2-surjective code can be found. The listing of these codes can be loaded from the web location http://www.sztaki.hu/~keri/codes-hu/lemez/Binary/K_9_2_classif.txt.

III. Ternary covering codes

Theorem 6: $K_3(n, R) = 3$ if and only if $\lfloor \frac{2n}{3} \rfloor \leq R < n$, and the number of inequivalent optimal covering codes is the coefficient of $x^{3R-2n+2}$ in the expansion of

$$\frac{1}{(1-x)^3(1-x^2)(1-x^3)} =$$

$$= 1 + 3x + 7x^2 + 14x^3 + 25x^4 + 41x^5 + \ldots$$

The sequence of coefficients in this expansion is A057524 in the on-line encyclopedia of integer sequences [21].

Proof: According to Theorem 1, optimal codes are those 3-word codes, for which there are at least $3n-3R-2$ coordinate positions containing each of the symbols $0, 1, 2$ in the codewords. The codewords may contain any of the three possible symbols in the remaining $n' = 3R - 2n + 2$ coordinate positions. Consequently, the number of inequivalent optimal codes is equal to the number of inequivalent $3 \times n'$ matrices with elements from $Z_3 = \{0, 1, 2\}$ up to row and column permutations, provided $n' > 0$. (For $n' = 0$, the optimal covering code is unique, from which the assertion of the theorem immediately follows for this special case.)

Now, consider the columns of an arbitrary $3 \times n'$ matrix with elements from $Z_3 = \{0, 1, 2\}$. If a column contains three identical symbols, then it can be assumed that these are identically 0s. In the case of two different symbols it can be assumed that there are two 0s and a single 1, but all of the three possible order of these should be taken into account. Finally, in the case of three different symbols it can be assumed that their order is always 0, 1, 2. Thus, our problem is reduced to the question of inequivalent $3 \times n'$ matrices built from

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

as their columns. Let n_i denote the number of occurrences of the above listed five columns. Here n_1 and n_5 can be arbitrary, but to exclude equivalent structures, we have to restrict for $n_2 \geq n_3 \geq n_4$. Consequently, the requested number of inequivalent matrices agrees with the number of integer solutions of the system

$$\begin{aligned} &n_1 + n_2 + n_3 + n_4 + n_5 = n', \\ &n_1 \geq 0, \\ &n_2 \geq n_3 \geq n_4 \geq 0, \\ &n_5 \geq 0, \end{aligned}$$

which is the coefficient of n' in the expansion of $1/((1-x)^3(1-x^2)(1-x^3))$, according to Theorem 2. ■

The remaining known classification results for ternary covering codes (which are not covered by Theorem 6) are tabulated in Table II. These are partly combinatorial (for the six cases where the optimal code is unique), partly computational (for $K_3(6, 3) = 6$) results, again. The classification for $K_3(5, 1) = 27$ is a combination of combinatorial and computational steps.

About the normality of the classified ternary optimal codes. Looking at Table II it can be seen that there exist normal optimal covering codes for $K_3(5, 1) = 27$ and $K_3(6, 3) = 6$. By running a simple computer program, also it can be noticed that all 17 optimal codes for $K_3(5, 1) = 27$ are normal with respect to 1 or 2 coordinates, and 24 from the 28 optimal codes for $K_3(6, 3) = 6$ are normal with respect to 2, 3, 4, 5 or 6 coordinates. None of the optimal codes are normal for the other four tabulated cases. The most striking example is found at $K_3(4, 1) = 9$ where at least three more codewords are required for the construction of a normal code with the same covering radius.

Example 3. To demonstrate the variety of optimal code structures belonging to $K_3(6, 3) = 6$, consider the following three codes:

$$\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 1&1&1&1&1&0 \\ 1&1&1&1&1&2 \\ 2&2&2&2&2&1 \\ 2&2&2&2&2&2 \end{bmatrix} \quad \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&2 \\ 1&1&1&1&1&0 \\ 1&1&1&1&1&1 \\ 2&2&2&2&2&2 \end{bmatrix}$$

$$\begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&1&1&1 \\ 1&1&1&0&1&2 \\ 1&2&2&1&2&0 \\ 2&1&2&2&0&1 \\ 2&2&1&2&2&2 \end{bmatrix}$$

The first two codes have five identical columns and both are normal with respect to the corresponding five coordinates. The first (balanced) code is normal according to the last column, too, but it is not so for the second (non-balanced) code. The third code consists of six different columns, it is balanced and abnormal. (We note that the three other abnormal optimal codes are unbalanced and all of the other 27 optimal codes – arranged into similar matrix scheme – have at least two identical columns.) The listing of all optimal codes is at the web location http://www.sztaki.hu/~keri/codes-hu/lemez/Ternary/K3_6_3_classif.txt.

IV. q-ARY COVERING CODES

Theorem 7: ([6, Theorem 7.32]) $K_q(2, 1) = q$, and the number of inequivalent optimal covering codes agrees with the number of multisets of positive integers whose sum is q, which is the coefficient of x^q in the expansion of

$$\prod_{j=1}^{\infty} \frac{1}{1-x^j} = 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+\dots$$

The sequence of coefficients in this expansion is A000041 in the on-line encyclopedia of integer sequences [21].

Theorem 6 can be generalized for arbitrary q in the following sense.

Theorem 8: $K_q(n, R) = q$ if and only if $\left\lfloor \frac{(q-1)n}{q} \right\rfloor \leq R < n$, and the number of inequivalent optimal covering codes depends only on the value of $s = qR - (q-1)(n-1)$ and is equal to the number of inequivalent $q \times s$ matrices with elements from Z_q up to row and column permutations when $s > 0$. The optimal covering code is unique and is equivalent to the repetition code if $s = 0$.

Proof: The pair of inequalities given here is a mere rearrangement of the conditions to Theorem 1. The proof is essentially the same as the first paragraph in the proof of Theorem 6. ■

A few of the classification results in connection with Theorem 8 are tabulated in Table III. The numbers and sets of inequivalent objects for these cases were found by computer search. The known classification results not covered by Theorems 7 and 8 are tabulated in Table IV.

V. Mixed covering codes

The first two theorems of this section can be proved by elementary combinatorial considerations.

Theorem 9: $K_{3,2}(t,b,R) = 2$ if and only if $t + \lfloor \frac{b}{2} \rfloor \le R < t+b$, and the number of inequivalent optimal covering codes is $(t+1)(2R-2t-b+2)$.

Theorem 10: Let $q_1 > q_2 > \cdots > q_k = 2$. Then $K_{q_1,q_2,\ldots,q_k}(n_1,n_2,\ldots,n_k,R) = 2$ if and only if $\sum_{i=1}^{k-1} n_i + \lfloor \frac{n_k}{2} \rfloor \le R < \sum_{i=1}^{k} n_i$, and the number of inequivalent optimal covering codes is $(2R - 2\sum_{i=1}^{k-1} n_i - n_k + 2) \prod_{i=1}^{k-1}(n_i + 1)$.

The following two theorems will be proved by induction. As there are different possibilities for the completion of the inductive step, we apply two different alternatives, one for the proof of Theorem 11, and the other for Theorem 12.

Theorem 11: $K_{3,2}(1,2R,R) = 3$, and the unique optimal covering code is

$$\{ (0,0,\ldots,0), (1,1,\ldots,1), (2,1,\ldots,1)\}.$$

Proof:

It is evident (or follows from [10, Lemma 2] and Theorem 9) that the given code is optimal, and its covering radius is exactly R. To base the induction, first we prove the uniqueness for $R = 1$.

For an arbitrary optimal code, there should be three different values in the ternary coordinate of the codewords. Otherwise all of 00, 01, 10 and 11 should occur in the two binary coordinates, which is impossible for a three-word code. Thus any optimal code is equivalent to a code of form

$$\{ (0,0,0), (1,*,*), (2,*,*)\}$$

where an asterisk is used for any of 0 or 1. To cover the word $(0,1,1)$ with the given radius, we need a codeword of form $(*,1,1)$; it can be assumed that this codeword is $(1,1,1)$. To cover both of the words $(2,0,1)$ and $(2,1,0)$, the binary coordinates of the third codeword should be 00 or 11. For both cases, the resulting code is equivalent to the code given in the theorem, specialized for $R = 1$.

To turn to the inductive step, assume that the assertion of the theorem is true for a given R, and try to conclude that it is true also for $R+1$. Let D be an optimal code belonging to $K_{3,2}(1,2R+2,R+1) = 3$ and C be the code that is obtained by omitting the last two coordinates from the codewords of D. The covering radius of C cannot be greater than R, as otherwise there exists a word $x \in Z_3 Z_2^{2R}$ such that $d(x,C) \ge R+1$. The extension of x with a pair of binary coordinates which differs from the last two coordinates of all codewords of D leads to a word $y \in Z_3 Z_2^{2R+2}$, for which $d(y,D) \ge R+2$, a contradiction. As $K_{3,2}(1,2R,R) = 3$, the covering radius of C should be equal to R, consequently C is optimal, and then, it should be equivalent to a code having the given structure by the inductive assumption. This means that D should be equivalent to a code having the form

$$\{ (0,0,\ldots,0,0,0), (1,1,\ldots,1,*,*), (2,1,\ldots,1,*,*)\}.$$

The replacement of the four asterisks by binary values can be done in 16 different ways, but this number of possible variations can be reduced to 7 inequivalent cases. The analysis of the possible covering radii for each case proves that the covering radius is greater than $R+1$ in 6 from the 7 cases, and it is equal to $R+1$ only if all of the four asterisks are replaced by 1. ∎

It can be shown easily that the optimal covering codes studied in Theorem 11 are 2-normal with respect to all binary coordinate, but it is not 3-normal. To disprove 3-normality, consider a word $x \in Z_3 Z_2^{2R}$ so that the one ternary and $R+1$ binary coordinates contain 0, while the remaining $R-1$ binary coordinates contain 1.

Theorem 12: $K_{3,2}(3,2R-3,R) = 3$ if $R > 1$, and the unique optimal covering code is

$$\{ (0,0,0,0,\ldots,0), (1,1,1,1,\ldots,1), (2,2,2,1,\ldots,1)\}.$$

Proof: Our inductive step for the proof works only from $R = 4$ and $b = 5$, so we need it to be proven initially for $R = 2$ and $R = 3$. This can be done either by hand or by a computer program. As the proof for $R = 3$ by hand would be a little complicated and lengthy, so it is better to accept the computational proof for these initial settings of the parameters.

Again, let D be an optimal code belonging to $K_{3,2}(3,2R-3,R) = 3$ where $R \ge 4$ and let C be the code that is obtained by omitting any two binary coordinates from the codewords of D. Then, it can be shown analogously to the previous proof, that the covering radius of C is equal to $R-1$, and consequently C, a code with at $2R-5 \ge 3$ binary coordinates having identical binary symbols in each codewords, is equivalent to the code given in the theorem. This is possible only if D itself is equivalent to a similar code with $2R-3$ binary coordinates. ∎

It is easy to see again that the optimal code belonging to $K_{3,2}(3,2R-3,R) = 3$ is 2-normal, its 3-normality, however depends on the value of R, namely it is 3-normal for $R = 2$, but it is not so for $R \ge 3$. To disprove 3-normality, consider a word $x \in Z_3^3 Z_2^{2R-3}$ so that the three ternary and R binary coordinates contain 0, while the remaining $R-3$ binary coordinates contain 1.

For the summary of the remaining known classification results for mixed ternary/binary optimal covering codes see

n	R	$K_2(n,R)$	N	Ref.	$L_2(n,R)$	N
6	1	12	2	[19]	12	2
7	1	16	1	[23]	16	1
8	1	32	10	[19]	32	10
15	1	2048	5983	[18]	2048	5983
8	2	12	277	[19]	12	277
9	2	16	4	[19]	16	4
10	3	12	11481	[13]	12	11481
23	3	4096	1	[4]	4096	1

TABLE I

CLASSIFICATION RESULTS FOR BINARY COVERING CODES

n	R	$K_3(n,R)$	N	Ref.	$L_3(n,R)$	N
3	1	5	1	[17]	6	5
4	1	9	1	[5]	12	2
5	1	27	17	[20]	27	17
5	2	8	1	[2]	9	342
11	2	729	1	[4]	>729	
6	3	6	28	*NEW*	6	24

TABLE II

CLASSIFICATION RESULTS FOR TERNARY COVERING CODES

Tables V-VIII. The last table from this group contains only new classification results.

Our classification results for mixed quaternary/ternary/binary optimal covering codes are tabulated in Table IX where the values of $K_{4,3,2}(q,t,b,R)$ are placed at base level and the numbers of inequivalent optimal codes are indicated by superscripts. All of these classification results are new. The value of $K_{\cdot}(\ldots)$ for some cases with small parameter values were studied in [16].

The proof for all classification results regarding mixed covering codes that are shown in the tables are computational. For the case $K_{3,2}(1,4,1) = 8$, however, it was declared without proof in [14] that the number of inequivalent optimal covering codes is 2. We confirmed the correctness of this assertion by giving a computational proof for it.

For the listing of optimal codes that belong to some particular cases of classification results regarding mixed covering codes see the file at the web locations http://www.sztaki.hu/∼keri/codes-hu/lemez/Mixed/K_1_4_1_classif.txt and all other files whose filenames end with _classif.txt or _uniq.txt, furthermore from the files in the web folder http://www.sztaki.hu/∼keri/codes-hu/lemez/Mixed2.

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[1] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading (1976).

q	4	4	4	4	5	5
s	2	3	4	5	2	3
N	21	79	269	839	54	471

TABLE III

NUMBER OF INEQUIVALENT $q \times s$ MATRICES WITH ELEMENTS FROM Z_q

q	n	R	$K_q(n,R)$	N	Ref.	$L_q(n,R)$
4	3	1	8	1	*NEW*	>8
4	4	2	7	8	[10]	>7
5	3	1	13	1	*NEW*	>13

TABLE IV

CLASSIFICATION RESULTS FOR QUATERNARY AND QUINARY COVERING CODES

t	b	R	$K_{3,2}(t,b,R)$	N	Ref.	$L_{3,2}(t,b,R)$	N
1	4	1	8	2	[14]	10	14
1	5	1	16	120	*NEW*	16	10
1	6	1	24	2	*NEW*	>24	
2	1	1	4	2	*NEW*	4	1
2	2	1	6	1	*NEW*	7	1
2	3	1	12	23	*NEW*	12	9
2	4	1	20	1	*NEW*	>20	
3	1	1	9	4	[12]	10	49
3	2	1	16	3	*NEW*	16	1
3	3	1	24	2	*NEW*	>24	
4	1	1	18	5	*NEW*	18	1

TABLE V

CLASSIFICATION RESULTS FOR MIXED BINARY/TERNARY COVERING CODES WHERE $R = 1$

t	b	R	$K_{3,2}(t,b,R)$	N	Ref.	$L_{3,2}(t,b,R)$	N
1	5	2	6	155	*NEW*	6	40
1	6	2	8	6	*NEW*	10	
1	7	2	12	1	*NEW*	>12	
2	2	2	3	5	*NEW*	3	2
2	3	2	4	6	*NEW*	4	2
2	4	2	6	1	*NEW*	7	4
2	5	2	11	91	*NEW*	12	
3	2	2	5	1	*NEW*	5	1
3	3	2	8	15	*NEW*	8	6
3	4	2	13	1	*NEW*	>13	
4	1	2	6	11	*NEW*	6	10
4	2	2	10	4	*NEW*	11	12
5	1	2	12	1	*NEW*	>12	

TABLE VI

CLASSIFICATION RESULTS FOR MIXED BINARY/TERNARY COVERING CODES WHERE $R = 2$

t	b	R	$K_{3,2}(t,b,R)$	N	Ref.	$L_{3,2}(t,b,R)$	N
1	7	3	6	573	*NEW*	6	148
1	8	3	8	21	*NEW*	>8	
1	9	3	12	5	*NEW*	>12	
2	4	3	3	5	*NEW*	4	127
2	5	3	4	12	*NEW*	4	3
2	6	3	6	2	*NEW*	7	14
3	2	3	3	16	*NEW*	3	9
3	4	3	5	2	*NEW*	5	2
3	5	3	8	131	*NEW*	8	61
4	1	3	3	5	*NEW*	3	4
4	2	3	4	11	*NEW*	4	9
4	3	3	6	45	*NEW*	6	36
5	1	3	4	1	*NEW*	4	1
5	2	3	7	1	*NEW*	7	1
6	1	3	9	4	[12]	>9	

TABLE VII

CLASSIFICATION RESULTS FOR MIXED BINARY/TERNARY COVERING CODES WHERE $R = 3$

t	b	$K_{3,2}(t,b,4)$	N
1	9	6	1102
2	6	3	5
2	7	4	20
2	8	6	2
3	4	3	19
3	6	5	6
3	7	8	583
4	2	3	40
4	3	3	6
4	4	4	22
4	5	6	132
5	1	3	14
5	2	3	2
5	3	4	2
5	4	7	4
6	1	3	1
6	2	5	1
6	3	8	11
7	1	6	37

TABLE VIII

CLASSIFICATION RESULTS FOR MIXED BINARY/TERNARY COVERING CODES WHERE $R = 4$

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q	t	b	$R=1$	$R=2$	$R=3$
1	0	1	2^{2}		
1	0	2	4^{9}	2^{4}	
1	0	3	7^{17}	2^{2}	2^{6}
1	0	4	8^{1}	4^{19}	2^{4}
1	0	5	16^{8}	7^{441}	2^{2}
1	0	6		8^{1}	4^{36}
1	0	7			7^{2445}
1	0	8			8^{2}

q	t	b	$R=1$	$R=2$	$R=3$
1	1	0	3^{3}		
1	1	1	5^{2}	2^{4}	
1	1	2	8^{5}	3^{4}	2^{8}
1	1	3	12^{1}	5^{13}	2^{4}
1	1	4		8^{1089}	3^{4}
1	1	5			5^{38}
1	1	6			8^{10985}

q	t	b	$R=1$	$R=2$	$R=3$
1	2	0	6^{1}	3^{10}	
1	2	1	12^{770}	4^{23}	2^{6}
1	2	2	18^{5}	6^{61}	3^{21}
1	2	3		8^{1}	4^{71}
1	2	4			6^{189}
1	2	5			8^{1}

q	t	b	$R=1$	$R=2$	$R=3$
1	3	0	12^{1}	4^{1}	3^{25}
1	3	1		8^{938}	3^{4}
1	3	2			4^{2}
1	3	3			7^{4}
1	4	0		9^{68}	3^{3}
1	4	1			5^{1}
1	4	2			8^{6}
1	5	0			7^{68}

q	t	b	$R=1$	$R=2$	$R=3$
2	0	1	6^{4}	2^{3}	
2	0	2	8^{1}	4^{52}	2^{6}
2	0	3	16^{20}	6^{110}	2^{3}
2	0	4		8^{74}	4^{116}
2	0	5			6^{640}
2	0	6			8^{203}
2	1	0	8^{7}	3^{7}	
2	1	1	12^{1}	4^{2}	2^{6}
2	1	2		7^{15}	3^{11}
2	1	3		10^{1}	4^{6}
2	1	4			7^{137}
2	2	0	16^{1}	5^{3}	3^{25}
2	2	1		8^{5}	4^{184}
2	2	2			5^{4}
2	3	0		10^{1}	4^{14}
2	3	1			6^{33}

q	t	b	$R=1$	$R=2$	$R=3$
3	0	1	16^{34}	4^{1}	2^{4}
3	0	2		8^{305}	4^{247}
3	0	3			4^{3}
3	1	0		6^{10}	3^{14}
3	1	1		10^{11}	4^{26}
3	1	2			6^{267}
3	2	0			4^{3}
3	2	1			7^{15}
4	0	1		11^{2}	4^{8}
4	0	2			6^{8}
4	1	0			4^{1}

TABLE IX

CLASSIFICATION RESULTS FOR MIXED 4/3/2 COVERING CODES