Description of the algorithm

Let C be an MDS code whose generator matrix is the $k \times n$ matrix

$$\begin{pmatrix} 1 & & & & 1 & 1 & \dots & 1 \\ & 1 & & & 1 & a_{22} & \dots & a_{2,n-k} \\ & & \ddots & & \dots & \dots & \dots & \dots \\ & & & 1 & 1 & a_{k2} & \dots & a_{k,n-k} \end{pmatrix}. \tag{1}$$

Its $(k-1) \times (n-k-1)$ submatrix

$$A_0 = \begin{pmatrix} a_{22} & \dots & a_{2,n-k} \\ \dots & \dots & \dots \\ a_{k2} & \dots & a_{k,n-k} \end{pmatrix} \tag{2}$$

is supposed to be lexicographically ordered, i.e.

$$1 < a_{22} < \dots < a_{2,n-k} < \dots < a_{k,2} < \dots < a_{k,n-k}. \tag{3}$$

Then A (or A_0) is said to be a standard form of code C. As all information about A is contained in A_0, it is enough to discuss this smaller matrix.

Now, let us consider the transformations T_{1j}, T_{2j}, T_{3j}, and T_4, between different standard forms of C that we shall describe first in the terminology in connection with A and after that also in connection with A_0.

In general, $T_{ij} : A_0 \longrightarrow A_0'$ where

$$A_0' = \begin{pmatrix} a_{22}' & \dots & a_{2,n-k}' \\ \dots & \dots & \dots \\ a_{k2}' & \dots & a_{k,n-k}' \end{pmatrix}. \tag{4}$$

$T_{1\pi}$ ($\pi \in S_{k-1}$) permutes the rows of A_0:

$$a_{ij}' = a_{\pi(i)j}. \tag{5}$$

(The convention of lexicographical ordering allows us to bypass over the similar permutation of columns.)

T_{2s} $(s = 2, \dots, k)$ norms a row $s \in \{2, \dots, k\}$ to all 1-s, and after the norming, rows 1 and s of A are exchanged.

$$a'_{ij} = \begin{cases} a_{ij}^{-1} & \text{if} \quad i = s, \\ a_{sj}^{-1} a_{ij} & \text{if} \quad i \neq s. \end{cases} \tag{6}$$

T_{3t} $(t = 2, \dots, n-k)$ norms a column $t \in \{2, \dots, n-k\}$ to all 1-s, and after the norming, columns k+1 and $k+t$ of A are exchanged.

$$a'_{ij} = \begin{cases} a_{ij}^{-1} & \text{if} \quad j = t, \\ a_{it}^{-1} a_{ij} & \text{if} \quad j \neq t. \end{cases} \tag{7}$$

T_4 brings C to another canonical form. This is actually a change of basis in the terminology with connection to A, pivoting on the matrix element of A that is in row 1 and column $k+1$.

$$a'_{ij} = 1 - a_{ij}. \tag{8}$$

I will be used for the identical transformation.

Referring to the column vectors of A as

$$e_1, e_2, \dots, e_k, a_1, a_2, \dots, a_{n-k}, \tag{9}$$

$T_{1\pi}$ and T_{2s} permutes only the rows of A. This can be interpreted also as the permutation of the unit vector columns of A. As regards T_{3t}, it permutes the non-unit vectors of A.

When multiplying $T_{i_1 j_1}$ with $T_{i_2 j_2}$ in the order that $T_{i_1 j_1}$ is applied first, and $T_{i_2 j_2}$ after $T_{i_1 j_1}$, then we shall use the expression $T_{i_1 j_1} T_{i_2 j_2}$ for this compound transformation, i.e. the order of applying the transformations is from left to right.

Taking – with some restrictions – different product expressions built from $T_{1\pi}, T_{2s}, T_{3t}$ and T_4, all codes that are PGL-equivalent to C and that are in standard form can be generated.

A transformation between PGL-equivalent standard forms can be regarded also as a re-ordering of the column vectors of A, but in an ordered list of column vectors only the vectors that stand in columns 1 to $k+1$ are relevant. The order of columns $k+1$ to n are always uniquely determined by the lexicographical rule.

Example. Let us see the correspondence between the transformations and the ordering of column vectors for $k = 2$.

transformation		first 3 elements in the ordered set
I	$\longleftrightarrow$	(e_1, e_2, a_1)
T_{22}	$\longleftrightarrow$	(e_2, e_1, a_1)
T_{3u} $(2 \leq u \leq n-k)$	$\longleftrightarrow$	(e_1, e_2, a_u)
T_4	$\longleftrightarrow$	(a_1, e_2, e_1)

Table 1 contains the listing of expressions for all possible case of column arrangement.

When $k = 2$ then there is nothing to do with transformations of type $T_{1\pi}$. For greater values of k, the second subscript of $T_{1\pi}$ as a permutation runs over $(k-1)!$ different values. For sake of convenience and also for the practical purpose of building an algorithm to find the different transforms, we propose to decompose the method into 2 modules. In one of these modules, the order of the rows of A_0 (i.e. the order of columns $e_2, \ldots e_k$ of A) is not cared with, while the other module of the algorithm (that is much simpler than the first one) deals only with the permutation of the rows for all different cases that were got in module 1.

We shall discuss only the first module of the algorithm in general. As it was mentioned already, there is a correspondence between the PGL-equivalent standard forms and the re-ordering of the column vectors of A where only the vectors that stand in columns 1 to $k+1$ are relevant. As the transformations of type $T_{1\pi}$ are left to a separate module of the algorithm, now the order of the vectors that stand in columns 2 to k are indifferent, too. The roles of columns 1 and $k+1$ still remain significant, because they belong to the row or column of A that contain all 1-s.

Though the permutation of the rows are neglected in this module, still we shall deal with row permutations, but only for the purpose of reducing the number of cases.

To turn to the details, let us consider structures of form (x, S, y) where S should be dealt with as an ordered set of $k-1$ vectors until the arithmetics is completed for a given case. As we are going to determine only one set S out of the sets that are built of the same elements (but in different order), for the sake of classification algoritm, S finally will be considered unordered. This is the motivation of applying braces when listing the elements of specific sets S. The ordering of S is necessary, however, for the arithmetics of the transformations to get correct results.

Clearly, the leading rows of Table 1 can be extended to the general case as

transformation		column vectors in (x, S, y)
I	$\longleftrightarrow$	$(e_1, \{e_2, e_3, \ldots, e_k\}, a_1)$
T_{22}	$\longleftrightarrow$	$(e_2, \{e_1, e_3, \ldots, e_k\}, a_1)$
T_{3u}	$\longleftrightarrow$	$(e_1, \{e_2, e_3, \ldots, e_k\}, a_u)$
T_4	$\longleftrightarrow$	$(a_1, \{e_2, e_3, \ldots, e_k\}, e_1)$
$T_{3u}T_{22}$	$\longleftrightarrow$	$(e_2, \{e_1, e_3, \ldots, e_k\}, a_u)$

In this manner, any case where $S \cup \{x, y\}$ contains all of $e_2, e_3, \ldots, e_k$ can be obtained from Table 1. Some other cases can be managed by referring to the symmetry in $e_2, e_3, \ldots, e_k$, e. g.

$$
\begin{array}{lll}
T_{23} & \longleftrightarrow & (e_3, \{e_1, e_2, e_4, \ldots, e_k\}, a_1) \\
T_{24} & \longleftrightarrow & (e_3, \{e_1, e_2, e_3, e_5, \ldots, e_k\}, a_1) \\
. & . & . \\
T_{3u}T_{23} & \longleftrightarrow & (e_3, \{e_1, e_2, e_4, \ldots, e_k\}, a_u) \\
T_{3u}T_{24} & \longleftrightarrow & (e_3, \{e_1, e_2, e_3, e_5, \ldots, e_k\}, a_u) \\
. & . & .
\end{array}
$$

A more sophisticated example is

$$T_{22}T_4T_{23} \quad \longleftrightarrow \quad (e_3, \{e_1, a_1, e_4, \ldots, e_k\}, e_2).$$

By pairing 2 different T_{2s} any other ways, $(k-1)(k-2)$ analogous cases can be managed, such as

$$
\begin{array}{lll}
T_{23}T_4T_{22} & \longleftrightarrow & (e_2, \{e_1, a_1, e_4, \ldots, e_k\}, e_3). \\
T_{22}T_4T_{24} & \longleftrightarrow & (e_4, \{e_1, a_1, e_3, e_5, \ldots, e_k\}, e_2). \\
T_{23}T_4T_{24} & \longleftrightarrow & (e_4, \{e_1, a_1, e_2, e_5, \ldots, e_k\}, e_3). \\
T_{24}T_4T_{22} & \longleftrightarrow & (e_2, \{e_1, a_1, e_3, e_5, \ldots, e_k\}, e_4). \\
T_{24}T_4T_{23} & \longleftrightarrow & (e_3, \{e_1, a_1, e_2, e_5, \ldots, e_k\}, e_4).
\end{array}
$$

All of these cases can be melted into 1 case by considering a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_{k-1}) \in S_{k-1}$ of the row indices $2, 3, \ldots, k$:

$$T_{2\pi_1}T_4T_{2\pi_2} \quad \longleftrightarrow \quad \big(e_{\pi_2}, \{e_1, a_1, e_{\pi_3}, \ldots, e_{\pi_{k-1}}\}, e_{\pi_1}\big).$$

Before building of a complete system of expressions (like in Table 1 for the specific case $k = 2$) that will cover the entire system of transformations between the standard MDS codes, first we put some preliminary remarks.

Let us consider a transformation F that is a product-expression having the form

$$F \longleftrightarrow (x, S, y) = \left(x, \left\{s_{\pi_1}, s_{\pi_2}, \ldots, s_{\pi_i}, \ldots, s_{\pi_{k-1}}\right\}, y\right).$$

Then

$$FT_4 \longleftrightarrow (y, S, x) = \left(y, \left\{s_{\pi_1}, s_{\pi_2}, \ldots, s_{\pi_i}, \ldots, s_{\pi_{k-1}}\right\}, x\right),$$

$$FT_{3v_j} \longleftrightarrow (x, S, a_{v_j}) = \left(x, \left\{s_{\pi_1}, s_{\pi_2}, \ldots, s_{\pi_i}, \ldots, s_{\pi_{k-1}}\right\}, a_{v_j}\right),$$

and

$$FT_{2\pi_i} \longleftrightarrow \left(s_{\pi_i}, \left\{s_{\pi_1}, s_{\pi_2}, \ldots, x, \ldots, s_{\pi_{k-1}}\right\}, y\right).$$

For different expressions F_1, F_2, their products in the two different orders are not equal in general, but there is a remarkable specific case when F_1 commutes with F_2, namely

$$T_{2\pi_i} T_{3v_j} = T_{3v_j} T_{2\pi_i}. \tag{10}$$

It can be mentioned that there are other identities with the product-expressions, e.g.

$$T_{2\pi_i} T_4 T_{2\pi_i} = T_4 T_{2\pi_i} T_4.$$

The product mark $\prod$ will be used in the meaning of

$$\prod_{i=l_1}^{l_2} F_i = F_{l_1} F_{l_1+1} \cdots F_{l_2},$$

always in this order of multiplication.

To avoid complications of the arithmetics, the following rules of thumb should be kept:

1) For a given j, T_{3v_j} is allowed to stand at most once in any product-expression.

2) $T_{2\pi_{i_1}} T_{2\pi_{i_2}}$ or $T_{3v_{j_1}} T_{3v_{j_2}}$ is not allowed as part of a product-expression.

3) The second rule has not to be eluded by applying the identity (10), i.e.

$T_{2\pi_{i_1}} T_{3v_j} T_{2\pi_{i_2}}$ or $T_{3v_{j_1}} T_{2\pi_i} T_{3v_{j_2}}$ is also not allowed in a product-expression.

To find the equivalent standard forms of MDS codes for arbitrary k, the following method turned to be a good idea: First products built from triplets such as $\prod_{i=1}^{j} T_{2\pi_i} T_4 T_{3v_i}$, $\prod_{i=1}^{j} T_4 T_{3v_i} T_{2\pi_i}$, $\prod_{i=1}^{j} T_{3v_i} T_{2\pi_i} T_4$ and $\prod_{i=1}^{j} T_{3v_i} T_4 T_{2\pi_i}$, and their sub-products are considered, the corresponding column-vector arrangement determined. Then expressions for any other possible arrangement are derived from the latter. All proofs are straightforward if the proper logical order of Theorems and their Corollaries is followed.

Theorem 1 *For* $2 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j-1} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_j} \longleftrightarrow \left(e_{\pi_j}, \left\{e_1, a_1, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j-1}}\right), \tag{11}$$

$$\left(\prod_{i=1}^{j-1} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_j} T_4 \longleftrightarrow \left(a_{v_{j-1}}, \left\{e_1, a_1, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right), \tag{12}$$

$$\left(\prod_{i=1}^{j} T_{2\pi_i} T_4 T_{3v_i}\right) \longleftrightarrow \left(a_{v_{j-1}}, \left\{e_1, a_1, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_j}\right). \tag{13}$$

Proof. Direct proof for $j = 2$, induction for $j > 2$. □

Corollary 1 *For* $2 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_1} \longleftrightarrow \left(e_1, \left\{a_{v_{j-1}}, a_1, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_j}\right), \tag{14}$$

$$\left(\prod_{i=1}^{j} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_1} T_4 \longleftrightarrow \left(a_{v_j}, \left\{a_{v_{j-1}}, a_1, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right). \tag{15}$$

Corollary 2 *For* $2 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_2} \longleftrightarrow \left(a_1, \left\{e_1, a_{v_{j-1}}, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_j}\right), \tag{16}$$

$$\left(\prod_{i=1}^{j} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_2} T_4 \longleftrightarrow \left(a_{v_j}, \left\{e_1, a_{v_{j-1}}, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right). \tag{17}$$

Corollary 3 *For* $2 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_2} T_4 T_{2\pi_1} \longleftrightarrow \left(e_1, \left\{a_{v_j}, a_{v_{j-1}}, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right), \tag{18}$$

$$\left(\prod_{i=1}^{j} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_1} T_4 T_{2\pi_2} \longleftrightarrow \left(a_1, \left\{a_{v_{j-1}}, a_{v_j}, a_{v_1}, \ldots, a_{v_{j-2}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right). \tag{19}$$

Corollary 4 *For* $3 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j-1} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_1} T_4 T_{2\pi_j} \longleftrightarrow \left(e_{\pi_j}, \left\{a_{v_{j-2}}, a_1, a_{v_1}, \ldots, a_{v_{j-3}}, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right), \tag{20}$$

$$\left(\prod_{i=1}^{j-1} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_2} T_4 T_{2\pi_j} \longleftrightarrow \left(e_{\pi_j}, \left\{e_1, a_{v_{j-2}}, a_{v_1}, \ldots, a_{v_{j-3}}, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right), \tag{21}$$

$$\left(\prod_{i=1}^{j-1} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_2} T_4 T_{2\pi_j} T_4 \longleftrightarrow \left(a_1, \left\{e_1, a_{v_{j-2}}, a_{v_1}, \ldots, a_{v_{j-3}}, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right). \tag{22}$$

$$\left(\prod_{i=1}^{j-1} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_2} T_4 T_{2\pi_j} T_4 T_{2\pi_1} \longleftrightarrow$$
$$\left(e_1, \left\{a_1, a_{v_{j-2}}, a_{v_1}, \ldots, a_{v_{j-3}}, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right). \tag{23}$$

Corollary 5 *For* $4 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j-2} T_{2\pi_i} T_4 T_{3v_i}\right) T_{2\pi_2} T_4 T_{2\pi_{j-1}} T_4 T_{2\pi_j} \longleftrightarrow \tag{24}$$
$$\left(e_{\pi_j}, \left\{e_1, a_{v_{j-3}}, a_{v_1}, \ldots, a_{v_{j-4}}, a_{v_{j-2}}, a_1, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right).$$

Theorem 2 *For* $1 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_4 T_{3v_i} T_{2\pi_i}\right) \longleftrightarrow \left(e_{\pi_j}, \left\{a_1, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_j}\right), \tag{25}$$

$$\left(\prod_{i=1}^{j} T_4 T_{3v_i} T_{2\pi_i}\right) T_4 \longleftrightarrow \left(a_{v_j}, \left\{a_1, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right), \tag{26}$$

$$\left(\prod_{i=1}^{j} T_4 T_{3v_i} T_{2\pi_i}\right) T_4 T_{3v_{j+1}} \longleftrightarrow \left(a_{v_j}, \left\{a_1, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right). \tag{27}$$

Proof. Direct proof for $j = 1$, induction for $j > 1$. □

Corollary 6 *For* $2 \le j \le k-1$

$$\left(\prod_{i=1}^{j-1} T_4 T_{3v_i} T_{2\pi_i}\right) T_4 T_{2\pi_j} \longleftrightarrow \left(e_{\pi_j}, \left\{a_1, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right). \tag{28}$$

Corollary 7 *For* $1 \le j \le k-1$

$$\left(\prod_{i=1}^{j} T_4 T_{3v_i} T_{2\pi_i}\right) T_4 T_{3v_{j+1}} T_{2\pi_1} \longleftrightarrow \left(a_1, \left\{a_{v_j}, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right), \tag{29}$$

$$\left(\prod_{i=1}^{j} T_4 T_{3v_i} T_{2\pi_i}\right) T_4 T_{3v_{j+1}} T_{2\pi_1} T_4 \longleftrightarrow \left(a_{v_{j+1}}, \left\{a_{v_j}, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right). \tag{30}$$

Corollary 8 *For* $2 \le j \le k-1$

$$\left(\prod_{i=1}^{j-1} T_4 T_{3v_i} T_{2\pi_i}\right) T_4 T_{3v_j} T_{2\pi_1} T_4 T_{2\pi_j} \longleftrightarrow \tag{31}$$

$$\left(e_{\pi_j}, \left\{a_{v_{j-1}}, a_{v_1}, \ldots, a_{v_{j-2}}, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right),$$

$$\left(\prod_{i=1}^{j-1} T_4 T_{3v_i} T_{2\pi_i}\right) T_4 T_{3v_j} T_{2\pi_1} T_4 T_{2\pi_j} T_4 \longleftrightarrow \tag{32}$$

$$\left(a_1, \left\{a_{v_{j-1}}, a_{v_1}, \ldots, a_{v_{j-2}}, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right).$$

Theorem 3 *For* $1 \le j \le k-1$

$$\left(\prod_{i=1}^{j-1} T_{3v_i} T_{2\pi_i} T_4\right) T_{3v_j} T_{2\pi_j} \longleftrightarrow \left(e_{\pi_j}, \left\{e_1, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_j}\right), \tag{33}$$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_{2\pi_i} T_4\right) \longleftrightarrow \left(a_{v_j}, \left\{e_1, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right), \tag{34}$$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_{2\pi_i} T_4\right) T_{3v_{j+1}} \longleftrightarrow \left(a_{v_j}, \left\{e_1, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right). \tag{35}$$

Proof. Direct proof for $j = 1$, induction for $j > 1$. □

Corollary 9 *For* $1 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_{2\pi_i} T_4\right) T_{2\pi_1} \longleftrightarrow \left(e_1, \left\{a_{v_j}, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right). \tag{36}$$

Corollary 10 *For* $2 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j-1} T_{3v_i} T_{2\pi_i} T_4\right) T_{2\pi_j} \longleftrightarrow \left(e_{\pi_j}, \left\{e_1, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right). \tag{37}$$

Corollary 11 *For* $1 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_{2\pi_i} T_4\right) T_{3v_{j+1}} T_{2\pi_1} \longleftrightarrow \left(e_1, \left\{a_{v_j}, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right), \tag{38}$$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_{2\pi_i} T_4\right) T_{3v_{j+1}} T_{2\pi_1} T_4 \longleftrightarrow \left(a_{v_{j+1}}, \left\{a_{v_j}, a_{v_1}, \ldots, a_{v_{j-1}}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right). \tag{39}$$

Corollary 12 *For* $2 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j-1} T_{3v_i} T_{2\pi_i} T_4\right) T_{3v_j} T_{2\pi_1} T_4 T_{2\pi_j} \longleftrightarrow \tag{40}$$

$$\left(e_{\pi_j}, \left\{a_{v_{j-1}}, a_{v_1}, \ldots, a_{v_{j-2}}, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right).$$

Theorem 4 *For* $1 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_4 T_{2\pi_i}\right) T_{3v_{j+1}} \longleftrightarrow \left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right), \tag{41}$$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_4 T_{2\pi_i}\right) T_{3v_{j+1}} T_4 \longleftrightarrow \left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right); \tag{42}$$

for $2 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_4 T_{2\pi_i}\right) \longleftrightarrow \left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right). \tag{43}$$

Proof. Direct proof for $j = 2$, induction for $j > 2$. $\square$

Corollary 13 *For* $1 \leq j \leq k-1$

$$\left(\prod_{i=1}^{j} T_{3v_i} T_4 T_{2\pi_i}\right) T_{3v_{j+1}} T_4 T_{3v_{j+2}} \longleftrightarrow \left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right). \tag{44}$$

The results of Theorems 1-4 and Corollaries 1-13 together with several irregular cases of vector arrangements are summarized in Tables 2-4. All those cases are regarded irregular that can be derived from the left side of formulae (11)–(44) with such values of j for which the Theorems or Corollaries are not applicable. For example, the left side item of (41) applied for $j = 0$ gives practically the right side item of (38) for $j = 0$. However, the left side item of (18) for $j = 1$ (!) belongs to the right side item of (23) for $j = 2$ (!). Some other instances of irregular cases are given explicitly in Tables 4.2–4.4.

Now, a detailed discussion for each case of possible vector arrangements will be presented – using the numbering that is applied in the tables.

The notation is slightly modified as regards the indexing of the a_{v_i}-s because it is more convenient through the discussions to denote the number of vectors other than e_{π_i}-s standing in the middle block of the arrangement by j. Then the number of e_{π_i}-s in the middle block is always $k-1-j$. This shifting of the subscript at v in certain cases should be taken into account when eqs. (11)– (44) are referred to during the discussions.

Case 1: $\left(e_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$.

Applicable for $0 \leq j \leq k-1$.

Transformation:

I for $j = 0$,

$T_{2\pi_1} T_4 T_{3v_1} T_4 T_{2\pi_1}$ for $j = 1$,

left of (18) for $j \geq 2$.

Loops for the algorithm:

$2 \leq \pi_1 < \pi_2 < \cdots < \pi_j \leq k$,

$2 \leq v_1 < v_2 < \cdots < v_j \leq n-k$.

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j} \cdot \binom{l}{j}$, where $l = n - k - 1$.

Number of instances after the permutation of the vectors of the middle block:

$\binom{k-1}{j} \cdot \binom{l}{j} \cdot (k-1)!$.

Case 2: $\left(e_1, \left\{a_{v_1}, \dots, a_{v_j}, e_{\pi_{j+1}}, \dots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$.

Applicable for $1 \le j \le k-1$.

Transformation: left of (36) for $j \ge 1$.

Loops for the algorithm:

$2 \le \pi_1 < \pi_2 < \cdots < \pi_{j-1} \le k$,

$\pi_j \in \{2, 3, \dots, k\} \setminus \{\pi_1, \pi_2, \dots, \pi_{j-1}\}$,

$2 \le v_1 < v_2 < \cdots < v_j \le n-k$.

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j-1} \cdot (k-j) \cdot \binom{l}{j} = j \cdot \binom{k-1}{j} \cdot \binom{l}{j}$.

Number of instances after the permutation of the vectors of the middle block:

$j \cdot \binom{k-1}{j} \cdot \binom{l}{j} \cdot (k-1)!$.

Case 3: $\left(e_1, \left\{a_{v_1}, \dots, a_{v_j}, e_{\pi_{j+1}}, \dots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$.

Applicable for $0 \le j \le k-1$.

Transformation:

left of (41) with $j = 0$ for $j = 0$,

left of (38) for $j \ge 1$.

Loops for the algorithm:

$2 \le \pi_1 < \pi_2 < \cdots < \pi_j \le k$,

$2 \le v_1 < v_2 < \cdots < v_j \le n-k$,

$v_{j+1} \in \{2, 3, \ldots, n-k\} \setminus \{v_1, v_2, \ldots, v_j\}$.

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j} \cdot \binom{l}{j} \cdot (l-j) = (j+1) \cdot \binom{k-1}{j} \cdot \binom{l}{j+1}$.

Number of instances after the permutation of the vectors of the middle block:

$(j+1) \cdot \binom{k-1}{j} \cdot \binom{l}{j+1} \cdot (k-1)!$.

Case 4: $\left(e_1, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$.

Applicable for $1 \le j \le k-1$.

Transformation:

left of (20) with $j = 1$ for $j = 1$,

left of (18) with $j = 1$ for $j = 2$,

left of (23) for $j \ge 3$.

Loops for the algorithm:

$2 \le \pi_1 < \pi_2 < \cdots < \pi_{j-1} \le k$,

$\pi_j \in \{2, 3, \ldots, k\} \setminus \{\pi_1, \pi_2, \ldots, \pi_{j-1}\}$,

$2 \le v_2 < v_3 < \cdots < v_j \le n-k$.

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j-1} \cdot (k-j) \cdot \binom{l}{j-1} = j \cdot \binom{k-1}{j} \cdot \binom{l}{j-1}$.

Number of instances after the permutation of the vectors of the middle block:

$j \cdot \binom{k-1}{j} \cdot \binom{l}{j-1} \cdot (k-1)!$.

Case 5: $\left(e_1, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$.

Applicable for $1 \le j \le k-1$.

Transformation: left of (14) for $j \ge 1$.

Loops for the algorithm:

$2 \le \pi_1 < \pi_2 < \cdots < \pi_j \le k$,

$2 \le v_2 < v_3 < \cdots < v_j \le n-k$,

$v_{j+1} \in \{2, 3, \ldots, n-k\} \setminus \{v_2, v_3, \ldots, v_j\}$.

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j} \cdot \binom{l}{j-1} \cdot (l-j+1) = j \cdot \binom{k-1}{j} \cdot \binom{l}{j}$.

Number of instances after the permutation of the vectors of the middle block:

$j \cdot \binom{k-1}{j} \cdot \binom{l}{j} \cdot (k-1)!$.

Case 6: $\left(a_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$.

Applicable for $0 \le j \le k-1$.

Transformation:

left of (26) with $j=0$ for $j=0$,

$T_{2\pi_1} T_4 T_{3v_1} T_4 T_{2\pi_1} T_4$ for $j=1$,

left of (19) for $j \ge 2$.

Loops and number of instances: see at Case 1.

Case 7: $\left(a_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$.

Applicable for $1 \le j \le k-1$.

Transformation:

left of (31) with $j=1$ for $j=1$,

left of (32) for $j \ge 2$.

Loops and number of instances: see at Case 2.

Case 8: $\left(a_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$.

Applicable for $0 \le j \le k-1$.

Transformation:

left of (27) with $j = 0$ for $j = 0$,

left of (29) for $j \geq 1$.

Loops and number of instances: see at Case 3.

Case 9: $\left(a_1, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation:

left of (12) with $j = 1$ for $j = 1$,

left of (21) with $j = 2$ for $j = 2$,

left of (22) for $j \geq 3$.

Loops and number of instances: see at Case 4.

Case 10: $\left(a_1, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation:

left of (13) with $j = 1$ for $j = 1$,

left of (16) for $j \geq 2$.

Loops and number of instances: see at Case 5.

Case 11: $\left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation:

left of (43) with $j = 1$ for $j = 1$,

left of (40) for $j \geq 2$.

Loops and number of instances: see at Case 2.

Case 12: $\left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation:

$T_4 T_{3v_1} T_4 T_{2\pi_1}$ for $j = 1$,

left of (31) for $j \geq 2$.

Loops and number of instances: see at Case 2.

Case 13: $\left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right)$.

Applicable for $2 \leq j \leq k-1$.

Transformation: left of (43) for $j \geq 2$.

Loops for the algorithm:

$2 \leq \pi_1 < \pi_2 < \cdots < \pi_{j-2} \leq k$,

$\pi_{j-1} \in \{2, 3, \ldots, k\} \setminus \{\pi_1, \pi_2, \ldots, \pi_{j-2}\}$,

$\pi_j \in \{2, 3, \ldots, k\} \setminus \{\pi_1, \pi_2, \ldots, \pi_{j-1}\}$,

$2 \leq v_1 < v_2 < \cdots < v_j \leq n-k$.

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j-2} \cdot (k-j+1) \cdot (k-j) \cdot \binom{l}{j} = j \cdot (j-1) \cdot \binom{k-1}{j} \cdot \binom{l}{j}$.

Number of instances after the permutation of the vectors of the middle block:

$j \cdot (j-1) \cdot \binom{k-1}{j} \cdot \binom{l}{j} \cdot (k-1)!$.

Case 14: $\left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation: left of (41) for $j \geq 1$.

Loops for the algorithm:

$2 \leq \pi_1 < \pi_2 < \cdots < \pi_{j-1} \leq k$,

$\pi_j \in \{2, 3, \ldots, k\} \setminus \{\pi_1, \pi_2, \ldots, \pi_{j-1}\}$,

$2 \le v_1 < v_2 < \cdots < v_j \le n-k,$

$v_{j+1} \in \{2, 3, \ldots, n-k\} \setminus \{v_1, v_2, \ldots, v_j\}.$

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j-1} \cdot (k-j) \cdot \binom{l}{j} \cdot (l-j) = j \cdot (j+1) \cdot \binom{k-1}{j} \cdot \binom{l}{j+1}.$

Number of instances after the permutation of the vectors of the middle block:

$j \cdot (j+1) \cdot \binom{k-1}{j} \cdot \binom{l}{j+1} \cdot (k-1)!.$

Case 15: $\left(e_{\pi_j}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right).$

Applicable for $1 \le j \le k-1$.

Transformation:

left of (11) with $j = 1$ for $j = 1$,

$T_{2\pi_1} T_4 T_{3v_2} T_4 T_{2\pi_2}$ for $j = 2$,

left of (21) for $j \ge 3$.

Loops and number of instances: see at Case 4.

Case 16: $\left(e_{\pi_j}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right).$

Applicable for $2 \le j \le k-1$.

Transformation: left of (37) for $j \ge 2$.

Loops for the algorithm:

$$2 \le \pi_1 < \pi_2 < \cdots < \pi_{j-2} \le k,$$

$$\pi_{j-1} \in \{2, 3, \ldots, k\} \setminus \{\pi_1, \pi_2, \ldots, \pi_{j-2}\},$$

$$\pi_j \in \{2, 3, \ldots, k\} \setminus \{\pi_1, \pi_2, \ldots, \pi_{j-1}\},$$

$$2 \le v_2 < v_3 < \cdots < v_j \le n - k.$$

Number of instances for a given j without the permutation of the vectors of the middle block:

$$\binom{k-1}{j-2} \cdot (k - j + 1) \cdot (k - j) \cdot \binom{l}{j-1} = j \cdot (j - 1) \cdot \binom{k-1}{j} \cdot \binom{l}{j-1}.$$

Number of instances after the permutation of the vectors of the middle block:

$$j \cdot (j - 1) \cdot \binom{k-1}{j} \cdot \binom{l}{j-1} \cdot (k - 1)!.$$

Case 17: $\left(e_{\pi_j}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$.

Applicable for $1 \le j \le k - 1$.

Transformation: left of (33) for $j \ge 1$.

Loops for the algorithm:

$$2 \le \pi_1 < \pi_2 < \cdots < \pi_{j-1} \le k,$$

$$\pi_j \in \{2, 3, \ldots, k\} \setminus \{\pi_1, \pi_2, \ldots, \pi_{j-1}\},$$

$$2 \le v_2 < v_3 < \cdots < v_j \le n - k,$$

$$v_{j+1} \in \{2, 3, \ldots, n - k\} \setminus \{v_2, v_3, \ldots, v_j\}.$$

Number of instances for a given j without the permutation of the vectors of the middle block:

$$\binom{k-1}{j-1} \cdot (k - j) \cdot \binom{l}{j-1} \cdot (l - j + 1) = j^2 \cdot \binom{k-1}{j} \cdot \binom{l}{j}.$$

Number of instances after the permutation of the vectors of the middle block:

$$j^2 \cdot \binom{k-1}{j} \cdot \binom{l}{j} \cdot (k - 1)!.$$

Case 18: $\left(e_{\pi_j}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$.

Applicable for $1 \le j \le k-1$.

Transformation:

left of (28) with $j = 1$ for $j = 1$,

left of (19) with $j = 1$ for $j = 2$,

left of (20) for $j \ge 3$.

Loops and number of instances: see at Case 4.

Case 19: $\left(e_{\pi_j}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right)$.

Applicable for $2 \le j \le k-1$.

Transformation: left of (28) for $j \ge 2$.

Loops and number of instances: see at Case 16.

Case 20: $\left(e_{\pi_j}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$.

Applicable for $1 \le j \le k-1$.

Transformation: left of (25) for $j \ge 1$.

Loops and number of instances: see at Case 17.

Case 21: $\left(e_{\pi_j}, \left\{e_1, a_1, a_{v_3}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right)$.

Applicable for $2 \le j \le k-1$.

Transformation:

left of (19) with $j = 0$ for $j = 2$,

$T_{2\pi_1} T_4 T_{3v_3} T_{2\pi_2} T_4 T_{2\pi_3}$ for $j = 3$,

left of (24) for $j \ge 4$.

Loops for the algorithm:

$2 \le \pi_1 < \pi_2 < \cdots < \pi_{j-2} \le k,$

$\pi_{j-1} \in \{2, 3, \dots, k\} \setminus \{\pi_1, \pi_2, \dots, \pi_{j-2}\},$

$\pi_j \in \{2, 3, \dots, k\} \setminus \{\pi_1, \pi_2, \dots, \pi_{j-1}\},$

$2 \le v_3 < v_4 < \cdots < v_j \le n - k.$

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j-2} \cdot (k - j + 1) \cdot (k - j) \cdot \binom{l}{j-2} = j \cdot (j - 1) \cdot \binom{k-1}{j} \cdot \binom{l}{j-2}.$

Number of instances after the permutation of the vectors of the middle block:

$j \cdot (j - 1) \cdot \binom{k-1}{j} \cdot \binom{l}{j-2} \cdot (k - 1)!.$

Case 22: $\left(e_{\pi_j}, \left\{e_1, a_1, a_{v_3}, \dots, a_{v_j}, e_{\pi_{j+1}}, \dots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right).$

Applicable for $2 \le j \le k - 1$.

Transformation: left of (11) for $j \ge 2$.

Loops for the algorithm:

$2 \le \pi_1 < \pi_2 < \cdots < \pi_{j-1} \le k,$

$\pi_j \in \{2, 3, \dots, k\} \setminus \{\pi_1, \pi_2, \dots, \pi_{j-1}\},$

$2 \le v_3 < v_4 < \cdots < v_j \le n - k,$

$v_{j+1} \in \{2, 3, \dots, n - k\} \setminus \{v_3, v_4, \dots, v_j\}.$

Number of instances for a given j without the permutation of the vectors of the middle block:

$\binom{k-1}{j-1} \cdot (k - j) \cdot \binom{l}{j-2} \cdot (l - j + 2) = j \cdot (j - 1) \cdot \binom{k-1}{j} \cdot \binom{l}{j-1}.$

Number of instances after the permutation of the vectors of the middle block:

$j \cdot (j - 1) \cdot \binom{k-1}{j} \cdot \binom{l}{j-1} \cdot (k - 1)!.$

Case 23: $\left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$.

Applicable for $0 \le j \le k-1$.

Transformation:

left of (42) with $j = 0$ for $j = 0$,

left of (39) for $j \ge 1$.

Loops and number of instances: see at Case 3.

Case 24: $\left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$.

Applicable for $0 \le j \le k-1$.

Transformation:

$T_4 T_{3v_1} T_4$ for $j = 0$,

left of (30) for $j \ge 1$.

Loops and number of instances: see at Case 3.

Case 25: $\left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$.

Applicable for $1 \le j \le k-1$.

Transformation: left of (42) for $j \ge 1$.

Loops and number of instances: see at Case 14.

Case 26: $\left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right)$.

Applicable for $0 \le j \le k-1$.

Transformation: left of (44) for $j \ge 0$.

Loops for the algorithm:

$2 \le \pi_1 < \pi_2 < \cdots < \pi_j \le k$,

$2 \le v_1 < v_2 < \cdots < v_j \le n-k$,

$v_{j+1} \in \{2, 3, \ldots, n-k\} \setminus \{v_1, v_2, \ldots, v_j\}$,

$v_{j+2} \in \{2, 3, \ldots, n-k\} \setminus \{v_1, v_2, \ldots, v_{j+1}\}$.

Number of instances for a given j without the permutation of the vectors of the middle block:

$$\binom{k-1}{j} \cdot \binom{l}{j} \cdot (l-j) \cdot (l-j-1) = (j+1) \cdot (j+2) \cdot \binom{k-1}{j} \cdot \binom{l}{j+2}.$$

Number of instances after the permutation of the vectors of the middle block:

$$(j+1) \cdot (j+2) \cdot \binom{k-1}{j} \cdot \binom{l}{j+2} \cdot (k-1)!.$$

Case 27: $\left(a_{v_{j+1}}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation:

$T_{2\pi_1} T_4 T_{3v_2} T_4$ for $j = 1$,

left of (17) for $j \geq 2$.

Loops and number of instances: see at Case 5.

Case 28: $\left(a_{v_{j+1}}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation: left of (34) for $j \geq 1$.

Loops and number of instances: see at Case 17.

Case 29: $\left(a_{v_{j+1}}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation: left of (35) for $j \geq 1$.

Loops for the algorithm:

$2 \leq \pi_1 < \pi_2 < \cdots < \pi_j \leq k$,

$2 \leq v_2 < v_3 < \cdots < v_j \leq n-k$,

$v_{j+1} \in \{2, 3, \ldots, n-k\} \setminus \{v_2, v_3, \ldots, v_j\}$,

$v_{j+2} \in \{2, 3, \ldots, n-k\} \setminus \{v_2, v_3, \ldots, v_{j+1}\}$.

Number of instances for a given j without the permutation of the vectors of the middle block:

$$\binom{k-1}{j} \cdot \binom{l}{j-1} \cdot (l-j+1) \cdot (l-j) = j \cdot (j+1) \cdot \binom{k-1}{j} \cdot \binom{l}{j+1}.$$

Number of instances after the permutation of the vectors of the middle block:

$$j \cdot (j+1) \cdot \binom{k-1}{j} \cdot \binom{l}{j+1} \cdot (k-1)!.$$

Case 30: $\left(a_{v_{j+1}}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation: left of (15) for $j \geq 1$.

Loops and number of instances: see at Case 5.

Case 31: $\left(a_{v_{j+1}}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation: left of (26) for $j \geq 1$.

Loops and number of instances: see at Case 17.

Case 32: $\left(a_{v_{j+1}}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right)$.

Applicable for $1 \leq j \leq k-1$.

Transformation: left of (27) for $j \geq 1$.

Loops and number of instances: see at Case 29.

Case 33: $\left(a_{v_{j+1}}, \left\{e_1, a_1, a_{v_3}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$.

Applicable for $2 \leq j \leq k-1$.

Transformation: left of (12) for $j \geq 2$.

Loops and number of instances: see at Case 22.

Case 34: $\left(a_{v_{j+1}}, \left\{e_1, a_1, a_{v_3}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right)$.

Applicable for $2 \leq j \leq k-1$.

Transformation: left of (13) for $j \geq 2$.

Loops for the algorithm:

$$2 \leq \pi_1 < \pi_2 < \cdots < \pi_j \leq k,$$

$$2 \leq v_3 < v_4 < \cdots < v_j \leq n-k,$$

$$v_{j+1} \in \{3, 4, \ldots, n-k\} \setminus \{v_3, v_4, \ldots, v_j\},$$

$$v_{j+2} \in \{3, 4, \ldots, n-k\} \setminus \{v_3, v_4, \ldots, v_{j+1}\}.$$

Number of instances for a given j without the permutation of the vectors of the middle block:

$$\binom{k-1}{j} \cdot \binom{l}{j-2} \cdot (l-j+2) \cdot (l-j+1) = j \cdot (j-1) \cdot \binom{k-1}{j} \cdot \binom{l}{j}.$$

Number of instances after the permutation of the vectors of the middle block:

$$j \cdot (j-1) \cdot \binom{k-1}{j} \cdot \binom{l}{j} \cdot (k-1)!.$$

Let us see, now, the counts for the number of cases from several aspects.

We have 34 global cases for the vector arrangement. Taking into account the multiplicity of the cases when different values of j are considered as different subcases, the count of this subcases will be as follows.

$k = 2$.

Cases 1, 3, 6, 8, 23–24 and 26 for $j = 0$ and 1 ($7 \cdot 2 = 14$ subcases);
cases 2,4–5, 7, 9–12, 14–15, 17–18, 20, 25, 27–32 for $j = 1$ (20 subcases).
The total number of subcases is 34.

$k \geq 3$.

The subcases listed for $k = 2$ (34 subcases);
all cases 1–34 for $j = 2, 3, k-1$ ($34 \cdot (k-2)$ subcases).

This way it has been found that the number of subcases is $34 \cdot (k-1)$ for any $k \geq 2$.

The count of the possible instances (i.e. the sum of the expressions given as the number of instances for each global case 1–34) has to be equal to

$$(l+k+1)\cdot(l+k)\cdot\binom{l+k-1}{k-1}, \tag{45}$$

because there are $l+k+1$ possibilities for the leading place in the vector arrangements, then remain $l+k$ possibilities for the ending place, while there are $\binom{l+k-1}{k-1}$ possibilities for the middle block. Consequently, if we take into account the all possible permutations of the elements that stand in the middle block, then we get

$$(l+k+1)\cdot(l+k)\cdot\binom{l+k-1}{k-1}\cdot(k-1)! = \frac{(l+k+1)!}{l!} \tag{46}$$

as the number of cases. This is the number of PGL-equivalent normal forms of a k-dimensional MDS code.

We can obtain the same result by actually summing the expressions that belong to cases 1–34 as the number of instances after the permutation of the vectors of the middle block. The number of terms in this sum can be reduced to 10 instead of 34, by merging the identical expressions:

Proposition 1

$$(k-1)!\sum_{j=0}^{k-1}\binom{k-1}{j}\cdot\Bigg[2\cdot\binom{l}{j}+8j\cdot\binom{l}{j}+4(j+1)\cdot\binom{l}{j+1}+4j\cdot\binom{l}{j-1}+$$

$$+2j(j-1)\cdot\binom{l}{j}+4j(j+1)\cdot\binom{l}{j+1}+4j(j-1)\cdot\binom{l}{j-1}+4j^2\cdot\binom{l}{j}+$$

$$+j(j-1)\cdot\binom{l}{j-2}+(j+1)(j+2)\cdot\binom{l}{j+2}\Bigg]=\frac{(l+k+1)!}{l!}.$$

Proof. The expression contained in the brackets can be brought to the simpler form

$$\binom{l+2}{j}\cdot(l+1)\cdot(l+2).$$

This reduces the sum in the left of the equation to

$$(k-1)!\cdot(l+1)\cdot(l+2)\cdot\sum_{j=0}^{k-1}\binom{k-1}{j}\cdot\binom{l+2}{j}=$$

$$=(k-1)!\cdot(l+1)\cdot(l+2)\cdot\binom{l+k+1}{k-1}=\frac{(l+k+1)!}{l!}.$$

□

TABLES

arrangement by length	
transformation	column order
I	(e_1, e_2, a_1)
T_{22}	(e_2, e_1, a_1)
T_{3u}	(e_1, e_2, a_u)
T_4	(a_1, e_2, e_1)
$T_{3u}T_{22}$	(e_2, e_1, a_u)
$T_{22}T_4$	(a_1, e_1, e_2)
$T_{3u}T_4$	(a_u, e_2, e_1)
T_4T_{22}	(e_2, a_1, e_1)
T_4T_{3u}	(a_1, e_2, a_u)
$T_{3u}T_{22}T_4$	(a_u, e_1, e_2)
$T_{22}T_4T_{22}$	(e_1, a_1, e_2)
$T_{22}T_4T_{3u}$	(a_1, e_1, a_u)
$T_{3u}T_4T_{22}$	(e_2, a_u, e_1)
$T_{3u}T_4T_{3v}$	(a_u, e_2, a_v)
$T_4T_{3u}T_{22}$	(e_2, a_1, a_u)
$T_4T_{3u}T_4$	(a_u, e_2, a_1)
$T_{3u}T_{22}T_4T_{22}$	(e_1, a_u, e_2)
$T_{3u}T_{22}T_4T_{3v}$	(a_u, e_1, a_v)
$T_{22}T_4T_{3u}T_{22}$	(e_1, a_1, a_u)
$T_{22}T_4T_{3u}T_4$	(a_u, e_1, a_1)
$T_{3u}T_4T_{3v}T_{22}$	(e_2, a_u, a_v)
$T_4T_{3u}T_{22}T_4$	(a_u, a_1, e_2)
$T_4T_{3u}T_4T_{22}$	(e_2, a_u, a_1)
$T_{3u}T_{22}T_4T_{3v}T_{22}$	(e_1, a_u, a_v)
$T_{22}T_4T_{3u}T_{22}T_4$	(a_u, a_1, e_1)
$T_{22}T_4T_{3u}T_4T_{22}$	(e_1, a_u, a_1)
$T_{3u}T_4T_{3v}T_{22}T_4$	(a_v, a_u, e_2)
$T_4T_{3u}T_{22}T_4T_{22}$	(a_1, a_u, e_2)
$T_4T_{3u}T_{22}T_4T_{3v}$	(a_u, a_1, a_v)
$T_{3u}T_{22}T_4T_{3v}T_{22}T_4$	(a_v, a_u, e_1)
$T_{22}T_4T_{3u}T_{22}T_4T_{22}$	(a_1, a_u, e_1)
$T_{3u}T_4T_{3v}T_{22}T_4T_{3w}$	(a_v, a_u, a_w)
$T_4T_{3u}T_{22}T_4T_{3v}T_{22}$	(a_1, a_u, a_v)
$T_4T_{3u}T_{22}T_4T_{3v}T_{22}T_4$	(a_v, a_u, a_1)

arrangement by column order	
transformation	column order
I	(e_1, e_2, a_1)
T_{3u}	(e_1, e_2, a_u)
$T_{22}T_4T_{22}$	(e_1, a_1, e_2)
$T_{22}T_4T_{3u}T_{22}$	(e_1, a_1, a_u)
$T_{3u}T_{22}T_4T_{22}$	(e_1, a_u, e_2)
$T_{22}T_4T_{3u}T_4T_{22}$	(e_1, a_u, a_1)
$T_{3u}T_{22}T_4T_{3v}T_{22}$	(e_1, a_u, a_v)
T_{22}	(e_2, e_1, a_1)
$T_{3u}T_{22}$	(e_2, e_1, a_u)
T_4T_{22}	(e_2, a_1, e_1)
$T_4T_{3u}T_{22}$	(e_2, a_1, a_u)
$T_{3u}T_4T_{22}$	(e_2, a_u, e_1)
$T_4T_{3u}T_4T_{22}$	(e_2, a_u, a_1)
$T_{3u}T_4T_{3v}T_{22}$	(e_2, a_u, a_v)
$T_{22}T_4$	(a_1, e_1, e_2)
$T_{22}T_4T_{3u}$	(a_1, e_1, a_u)
T_4	(a_1, e_2, e_1)
T_4T_{3u}	(a_1, e_2, a_u)
$T_{22}T_4T_{3u}T_{22}T_4T_{22}$	(a_1, a_u, e_1)
$T_4T_{3u}T_{22}T_4T_{22}$	(a_1, a_u, e_2)
$T_4T_{3u}T_{22}T_4T_{3v}T_{22}$	(a_1, a_u, a_v)
$T_{3u}T_{22}T_4$	(a_u, e_1, e_2)
$T_{22}T_4T_{3u}T_4$	(a_u, e_1, a_1)
$T_{3u}T_{22}T_4T_{3v}$	(a_u, e_1, a_v)
$T_{3u}T_4$	(a_u, e_2, e_1)
$T_4T_{3u}T_4$	(a_u, e_2, a_1)
$T_{3u}T_4T_{3v}$	(a_u, e_2, a_v)
$T_{22}T_4T_{3u}T_{22}T_4$	(a_u, a_1, e_1)
$T_4T_{3u}T_{22}T_4$	(a_u, a_1, e_2)
$T_4T_{3u}T_{22}T_4T_{3v}$	(a_u, a_1, a_v)
$T_{3u}T_{22}T_4T_{3v}T_{22}T_4$	(a_v, a_u, e_1)
$T_{3u}T_4T_{3v}T_{22}T_4$	(a_v, a_u, e_2)
$T_4T_{3u}T_{22}T_4T_{3v}T_{22}T_4$	(a_v, a_u, a_1)
$T_{3u}T_4T_{3v}T_{22}T_4T_{3w}$	(a_v, a_u, a_w)

Table 1. Different transformations for 2-dimensional MDS codes

	transformation	column arrangement	scope
1.	$\left(e_1, \left\{e_{\pi_1}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$	I	$k \geq 2$
	$\left(e_1, \left\{a_{v_1}, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$	$T_{2\pi_1} T_4 T_{3v_1} T_4 T_{2\pi_1}$	$k \geq 2$
	$\left(e_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$	(18) for $j \geq 2$	$k \geq 3$
2.	$\left(e_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$	(36) for $j \geq 1$	$k \geq 2$
3.	$\left(e_1, \left\{e_{\pi_1}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_1}\right)$	(41) for $j = 0$	$k \geq 2$
	$\left(e_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$	(38) for $j \geq 1$	$k \geq 2$
4.	$\left(e_1, \left\{a_1, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_1}\right)$	(20) for $j = 1$	$k \geq 2$
	$\left(e_1, \left\{a_1, a_{v_2}, e_{\pi_3}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_2}\right)$	(18) for $j = 1$	$k \geq 3$
	$\left(e_1, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$	(23) for $j \geq 3$	$k \geq 4$
5.	$\left(e_1, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$	(14) for $j \geq 1$	$k \geq 2$
6.	$\left(a_1, \left\{e_{\pi_1}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$	(26) for $j = 0$	$k \geq 2$
	$\left(a_1, \left\{a_{v_1}, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$	$T_{2\pi_1} T_4 T_{3v_1} T_4 T_{2\pi_1} T_4$	$k \geq 2$
	$\left(a_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$	(19) for $j \geq 2$	$k \geq 3$
7.	$\left(a_1, \left\{a_{v_1}, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_1}\right)$	(31) for $j = 1$	$k \geq 2$
	$\left(a_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$	(32) for $j \geq 2$	$k \geq 3$
8.	$\left(a_1, \left\{e_{\pi_1}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_1}\right)$	(27) for $j = 0$	$k \geq 2$
	$\left(a_1, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$	(29) for $j \geq 1$	$k \geq 2$
9.	$\left(a_1, \left\{e_1, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_1}\right)$	(12) for $j = 1$	$k \geq 2$
	$\left(a_1, \left\{e_1, a_{v_2}, e_{\pi_3}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_2}\right)$	(21) for $j = 2$	$k \geq 3$
	$\left(a_1, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$	(22) for $j \geq 3$	$k \geq 4$
10.	$\left(a_1, \left\{e_1, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_2}\right)$	(13) for $j = 1$	$k \geq 2$
	$\left(a_1, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$	(16) for $j \geq 2$	$k \geq 3$
11.	$\left(e_{\pi_j}, \left\{a_{v_1}, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$	(43) for $j = 1$	$k \geq 2$
	$\left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$	(40) for $j \geq 2$	$k \geq 3$

Table 2. Different transformations for k-dimensional MDS codes (Part 1)

	transformation	column arrangement	scope
12.	$\left(e_{\pi_1}, \left\{a_{v_1}, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$ $\left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$	$T_4T_{3v_1}T_4T_{2\pi_1}$ (31) for $j \geq 2$	$k \geq 2$ $k \geq 3$
13.	$\left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right)$	(43) for $j \geq 2$	$k \geq 3$
14.	$\left(e_{\pi_j}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$	(41) for $j \geq 1$	$k \geq 2$
15.	$\left(e_{\pi_1}, \left\{e_1, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$ $\left(e_{\pi_2}, \left\{e_1, a_{v_2}, e_{\pi_3}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$ $\left(e_{\pi_j}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$	(11) for $j = 1$ $T_{2\pi_1}T_4T_{3v_2}T_4T_{2\pi_2}$ (21) for $j \geq 3$	$k \geq 2$ $k \geq 3$ $k \geq 4$
16.	$\left(e_{\pi_j}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right)$	(37) for $j \geq 2$	$k \geq 3$
17.	$\left(e_{\pi_j}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$	(33) for $j \geq 1$	$k \geq 2$
18.	$\left(e_{\pi_1}, \left\{a_1, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$ $\left(e_{\pi_2}, \left\{a_1, a_{v_2}, e_{\pi_3}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$ $\left(e_{\pi_j}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$	(28) for $j = 1$ (19) for $j = 1$ (20) for $j \geq 3$	$k \geq 2$ $k \geq 3$ $k \geq 4$
19.	$\left(e_{\pi_j}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right)$	(28) for $j \geq 2$	$k \geq 3$
20.	$\left(e_{\pi_j}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$	(25) for $j \geq 1$	$k \geq 2$
21.	$\left(e_{\pi_2}, \left\{e_1, a_1, e_{\pi_3}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_1}\right)$ $\left(e_{\pi_3}, \left\{e_1, a_1, a_{v_3}, e_{\pi_4}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_2}\right)$ $\left(e_{\pi_j}, \left\{e_1, a_1, a_{v_3}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_{j-1}}\right)$	(19) for $j = 0$ $T_{2\pi_1}T_4T_{3v_3}T_{2\pi_2}T_4T_{2\pi_3}$ (24) for $j \geq 4$	$k \geq 3$ $k \geq 4$ $k \geq 5$
22.	$\left(e_{\pi_j}, \left\{e_1, a_1, a_{v_3}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+1}}\right)$	(11) for $j \geq 2$	$k \geq 3$
23.	$\left(a_{v_1}, \left\{e_{\pi_1}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$ $\left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$	(42) for $j = 0$ (39) for $j \geq 1$	$k \geq 2$ $k \geq 2$
24.	$\left(a_{v_1}, \left\{e_{\pi_1}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$ $\left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$	$T_4T_{3v_1}T_4$ (30) for $j \geq 1$	$k \geq 2$ $k \geq 2$

Table 3. Different transformations for k-dimensional MDS codes (Part 2)

	transformation	column arrangement	scope
25.	$\left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$	(42) for $j \geq 1$	$k \geq 2$
26.	$\left(a_{v_{j+1}}, \left\{a_{v_1}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right)$	(44) for $j \geq 0$	$k \geq 2$
27.	$\left(a_{v_2}, \left\{e_1, e_{\pi_2}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$ $\left(a_{v_{j+1}}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_1\right)$	$T_{2\pi_1} T_4 T_{3v_2} T_4$ (17) for $j \geq 2$	$k \geq 2$ $k \geq 3$
28.	$\left(a_{v_{j+1}}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$	(34) for $j \geq 1$	$k \geq 2$
29.	$\left(a_{v_{j+1}}, \left\{e_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right)$	(35) for $j \geq 1$	$k \geq 2$
30.	$\left(a_{v_{j+1}}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_1\right)$	(15) for $j \geq 1$	$k \geq 2$
31.	$\left(a_{v_{j+1}}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$	(26) for $j \geq 1$	$k \geq 2$
32.	$\left(a_{v_{j+1}}, \left\{a_1, a_{v_2}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right)$	(27) for $j \geq 1$	$k \geq 2$
33.	$\left(a_{v_{j+1}}, \left\{e_1, a_1, a_{v_3}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, e_{\pi_j}\right)$	(12) for $j \geq 2$	$k \geq 3$
34.	$\left(a_{v_{j+1}}, \left\{e_1, a_1, a_{v_3}, \ldots, a_{v_j}, e_{\pi_{j+1}}, \ldots, e_{\pi_{k-1}}\right\}, a_{v_{j+2}}\right)$	(13) for $j \geq 2$	$k \geq 3$

Table 4. Different transformations for k-dimensional MDS codes (Part 3)