A short introduction to the theory of n-arcs and MDS codes

Let us start with a notice about the very close connection between n-arcs, MDS codes, superregular matrices and linearly independent sets in vector spaces – quoting the words of Hirshfeld [12].

> "The following four notions are equivalent for $n \geq k$:
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> 1. (CODING THEORY) a *maximum distance separable* (MDS) linear code C of length n, dimension k and hence minimum distance $d = n - k - 1$, that is, an $[n, k, n - k + 1]$ code over $\mathbf{F}_q$;
>
> 2. (MATRIX THEORY) a $k \times n - k$ matrix A with entries in $\mathbf{F}_q$ such that every minor is non-zero;
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> 3. (VECTOR SPACE) a set K' of n vectors in $V(k, q)$, the vector space of k dimensions over $\mathbf{F}_q$, with any k linearly independent;
>
> 4. (PROJECTIVE SPACE) an n-arc in $\mathrm{PG}(k - 1, q)$, that is, a set K of n points with at most $k - 1$ in any hyperplane of the projective space of $k - 1$ dimensions over $\mathbf{F}_q$."

Hirshfeld did not use the notion of 'superregular matrix', it was introduced by Roth and Lempel [27] for matrices with elements from $\mathrm{GF}(q)$ having the property that every square submatrix (minor) is non-singular. We might call these matrices also 'totally positive matrices' as it is often used for the similar property of real matrices.

MDS codes are very interesting constructions among linear codes. A linear code over a finite field $GF(q)$ of length n, dimension k and minimum distance d is called MDS (maximum distance separable) if it attains the Singleton bound $d = n - k + 1$.

MacWilliams and Sloane introduce MDS codes in the first sentence of Chapter 11 of [24] with the remarkable qualification as follows.

> "We come now to one of the most fascinating chapters in all of coding theory: MDS codes."

Any MDS code over $GF(q)$ of length n and dimension k is always a k-surjective code with minimum number of codewords, therefore, MDS codes play an important role in all areas where k-surjective codes are studied (error-correcting and covering codes, covering arrays etc.).

For more details see the cited works and, possibly, other items of the attached bibliography.