Exact solution approaches for bilevel lot-sizing

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Abstract

In this paper we propose exact solution methods for a bilevel uncapacitated lot-sizing problem with backlogs. This is an extension of the classical uncapacitated lot-sizing problem with backlogs, in which two autonomous and self-interested decision makers constitute a 2 echelon supply chain. The leader buys items from the follower in order to meet external demand at lowest cost. The follower also tries to minimize its costs. Both parties may backlog. We study the leader's problem, i.e., how to determine supply requests over time to minimize its costs in view of the possible actions of the follower. We develop two mixed-integer linear programming reformulations, as well as cutting planes to cut off feasible, but suboptimal solutions. We compare the reformulations on a series of benchmark instances.

Key words Inventory/Production planning with two echelons; Bilevel optimization; Integer Programming with cutting planes; Extended formulations

1 Introduction

This paper focuses on a bilevel lot-sizing model in a two level supply chain. The problem involves two self-interested decision makers, acting sequentially. The upper level, the *leader*, faces a time varying external demand that he wants to serve at a minimum cost. In order to meet the demand, he requests supplies from the lower level decision maker, the *follower*. The follower in turn tries to meet the requests at a minimum cost. Backlogging is possible at both levels. However, when the follower backlogs some of the requests, the backlogging cost is payed as a penalty to the leader, which reduces the leader's costs. While the follower's problem is a single-item uncapacitated lot-sizing problem with backlogging, the leader's problem is a more complex one, as it has to anticipate the late deliveries along with the corresponding penalties of the follower. Moreover, a late delivery from the follower my result in backlogging some of the leader's demands.

While most of the operations research literature investigates lot-sizing models with a single decision maker, it is widely recognized that the lot-sizing decisions of autonomous partners in the supply chain mutually affect each other. Recently, new approaches have been investigated to fill this gap: integrated models with the objective of minimizing the total cost by centralized planning [26], and coordination mechanisms for driving the self-interested partners towards optimal performance on the system level without giving up autonomy or data privacy [2, 9]. In this paper we pursue a different approach based on bilevel optimization.

In applications, the *leader* may correspond to a retailer who faces external demand. The *follower* may be a supplier of the retailer. Alternatively, in a production/distribution problem, hierarchical planning is carried out by different departments of an enterprise. The logistics department is the leader who faces the customers' demands, and the manufacturing department produces the final products at a minimum cost.

Illustrative example. There are $n = 10$ time periods. The leader's fixed production cost, and the marginal production, holding and backlogging costs are $f^1 = 100$, $p^1 = 1$, $h^1 = 6$, and $g^1 = 18$, respectively, in all periods. The cost parameters of the follower are $f^2 = 492$, $p^2 = 1$, $h^2 = 5$, and $g^2 = 6$. Table 1 presents an optimal solution for this instance of the bilevel lot-sizing problem. The columns of the table are indexed by the time periods 1 through 10. The rows depict the external demand d^1_t, the supply requests sent to the follower δ_t, the supply received from the follower x^1_t, and the stocking and backlogging quantities, s^1_t and r^1_t, respectively, of the leader; and the production plan x^2_t, the stocking and backlogging quantities, s^2_t and r^2_t, respectively, of the follower. Notice that the leader never backlogs in this example. Moreover, the external demand d^1_t in periods $t = 2, 7, 9$ is satisfied partly from production and partly from stock, i.e., $s^1_{t-1} + x^1_t = d^1_t$. This distinguishes the leader's problem from the uncapacitated lot-sizing problem, which always admits an optimal solution such that the demand of a period is uniquely served either from production, or from stock, or from backlog.

By a clever choice of the supply requests, the leader my prevent, or enforce backlogging at the follower. For instance, in time period 1, the leader inflates demand (82 instead of the external demand 71) in order to prevent the follower from backlogging, which would cause expensive late deliveries for the leader as well. To prevent backlogging, an amount of $f^2/g^2 = 492/6 = 82$ is needed. On the other hand, the leader moves some demand from period 6 to period 5. The supply requests for period 5 is then the maximum amount that does not trigger production at the follower. To summarize, by early demands the leader may obtain extra backlog compensation, which decreases its costs.

Main contributions and structure of the paper. We propose two mixed integer linear programming formulations for solving the bilevel lot-sizing problem. The two formulations differ in the modeling of the follower's optimality con-

Table 1: Optimal solution of a sample problem

t	1	2	3	4	5	6	7	8	9	10
d_t^1	71	84	43	21	4	81	59	44	32	46
δ_t	82	73	68		42.72	39.77	57.51	55.46	21.93	44.61
x_t^1	82	73	68			82.49	57.51	55.46	21.93	44.61
s_t^1	11		25	4		1.49		11.46	1.39	
r_t^1										
x_t^2	82	141				140		122		
s_t^2		68					57.51	66.54	44.61	
r_t^2					42.72					

ditions. The first formulation is based on a polyhedral characterization of those demand sequences for an uncapacitated lot-sizing problem with backlogs that lead to an optimal solution with pre-specificed structure. The second model is derived from a shortest path formulation of the follower's problem. We also provide new inequalities to cut off suboptimal solutions of uncapacitated lot-sizing problems with backlogs. To the best of our knowledge, this is the first attempt for solving bilevel lot-sizing problems to optimality. Moreover, the technique of expressing the follower's optimality conditions without referring to complementarity conditions is new.

Structure of the paper. The related literature is surveyed in Section 2. The necessary background in lot-sizing with backlogs is recapitulated in Section 3. The bilevel lot-sizing problem is formally defined in Section 4. The two MIP models are presented in Section 5. The new cutting planes for cutting off suboptimal solutions are derived in Section 6. Finally, the approaches are assessed in computation experiments in Section 7, and conclusions are drawn in Section 8.

2 Related literature

2.1 Lot-sizing

Fundamental results on dynamic lot-sizing models were published by [41], and [43]. These papers consider uncapacitated lot-sizing models where the deterministic, time varying demand is known in advance over a finite planning horizon. Over the past decades the basic models have been extended by production capacities and various side constraints, for an overview see e.g., [3, 30, 33]. Albeit dynamic programming is still the most efficient method for solving the tractable cases [29, 40, 41, 43], they have been complemented by linear programming formulations for describing the convex hull of feasible solutions, see e.g., [4, 5, 31, 32, 34, 39, 25]. Many times, it is easier to work with *extended formulations*, when new variables and constraints are introduced to obtain the linear formulation. The modeling of various features in lot-sizing by mixed-integer programs (MIP) are investigated in e.g., [7, 11]. As further extensions, different lot-sizing and scheduling models, including small-bucket and large-bucket, discrete and continuous time formulations, as well as single- and multi-level models are presented in [16, 30].

2.2 Lot-sizing in supply chains

The need for studying the interacting lot-sizing decisions of multiple autonomous parties in a supply chain is widely recognized. One of the possible approaches is *integration*, when the different parties jointly solve the interrelated planning problems, see e.g., [26] for an overview, and [1, 19, 29] for applications. Important recent results in integrated planning include the work of [19], who investigated the case of serial chains, constrained capacities, and concave cost functions, and introduced a dynamic program whose running time is polynomial when the number of levels in the chain is fixed. A dynamic program that runs in $O(n^2 \log n)$ and a tight extended formulation is presented by [29] for uncapacitated two-level lot-sizing, and a formulation is derived for the multi-item, multi-client case. [20] developed efficient methods for solving the integrated production and transportation planning problem under various assumptions.

A drawback of integration is the mutual sharing of all the planning relevant information, which is sometimes unrealistic. A game theoretic approach alleviates this burden by using coordination mechanisms between the parties to drive the supply chain towards a system-wide optimal performance [2, 9]. The decentralized planning, integrated, coordinated, and bilevel approaches to the same lot-sizing problem in a two-player supply chain are compared in [23]. In particular, an enumeration based method is proposed for the bilevel lot-sizing problem with backlogs, but it works only on problem instances with at most 10 time periods.

2.3 Bilevel programming

Bilevel programming addresses decision and optimization problems whose outcome is determined by the interplay of two self-interested decision makers who decide sequentially. The first decision maker, the so-called *leader*, is assumed to have a complete knowledge of the second decision maker's, the *follower's* problem and parameters. Therefore, to optimize its own objective function, the leader must consider the response that it can expect from the follower. Bilevel optimization problems usually have two variants. In the *optimistic case*, if the follower has multiple optimal solutions, the leader can pick one which is the most advantageous for him. The *pessimistic case* is just the opposite, i.e., if the follower has several optimal solutions for a set of parameters, the leader assumes that the least advantageous will be realized. The motivation for bilevel programming stems from economic game theory. In a two-player *Stackelberg game* two competing firms, the market leader and a follower company, for example a new entrant, produce equivalent goods. The firms decide their production quantities sequentially, which together determine the market price, with the aim of maximizing their own profit [37]. The basic modeling and solution techniques in bilevel programming are presented in [13]. A review of applicable solution methods for various classes of bilevel programs is given in [10], whereas reformulations of continuous bilevel optimization problems into single-level problems are discussed in [14]. A combinatorial perspective on bilevel problems is presented in [28]. Recent development in solution methods are presented in e.g., [15, 17, 38].

2.4 Related applications of bilevel programming

Despite the advances in generic solution methods, up to now, the literature of bilevel approaches to lot-sizing and planning problems in supply chains is rather scarce. The few works in this field include the paper of [12], where a supply chain of multiple parties is studied, and a heuristic solution method is proposed for finding locally optimal solution at each party. [27] investigate the production planning problem of a pharmaceutical company. [36] introduced a bilevel programming model to a production and distribution planning problem in a supply chain, where the follower's problem can be modeled by linear programs, whose parametric solutions can be computed efficiently. A similar production and distribution problem subject to uncertainties is formulated as a probabilistic bilevel problem in [35]. [42] investigate the problem of coordinated planning in a supply chain under hard service time requirements.

The application of bilevel programming to the coordination of multi-divisional organizations has been proposed in [6]. The upper level problem is that of the corporate unit, who wishes to set the internal transfer prices among the divisions in such a way that the local optimal decisions of the divisions coincide with the corporate optimum. Bilevel approaches to different production scheduling problems include [8, 21, 22, 24].

3 Background in uncapacitated lot-sizing with backlogs

In this section we recapitulate fundamental results on uncapacitated lot-sizing problems with backlogs (ULSB). The problem with a linear cost function can be stated as a mixed-integer linear program:

$$\min\left\{\sum_{t=1}^{n}(p_t x_t + f_t y_t + h_t s_t + g_t r_t) \mid (2) - (6)\right\} \tag{1}$$

where

$$x_t + (s_{t-1} - r_{t-1}) = \delta_t + (s_t - r_t), \quad t = 1, \ldots, n \tag{2}$$
$$x_t \leq M y_t, \quad t = 1, \ldots, n \tag{3}$$
$$s_0 = s_n = r_0 = r_n = 0, \tag{4}$$
$$x_t, s_t, r_t, \geq 0, \quad t = 1, \ldots, n \tag{5}$$
$$y_t \in \{0, 1\}, \quad t = 1, \ldots, n \tag{6}$$

In this formulation, p_t, f_t, h_t, r_t, and δ_t denote the marginal production cost, the fixed production cost, the marginal inventory holding cost, the marginal backlogging cost, and the demand in time period t, respectively; and $M = \sum_{t=1}^{n} \delta_t$ is a big constant. The variables x_t, s_t, and r_t represent the production, the stocking and backlogging quantities, respectively, and the binary variables y_t indicate whether there is production in period t or not. Let X^{BL} denote the set of feasible solutions of ULSB:

$$X^{BL} = \{(x, y, s, r) \in \mathbb{R}^n_+ \times \{0, 1\}^n \times \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \mid (x, y, s, r) \text{ satisfy (2)-(6)}\}.$$

Firstly, recall a basic structural property of ULSB:

Proposition 1 *([31]) The extreme points of* $\operatorname{conv}(X^{BL})$ *are of the following structure: there exists* $2q$ *indices* $1 = \ell_1 \leq i_1 < \ell_2 \leq i_2 < \cdots < \ell_q \leq i_q \leq n$ *such that*

- $x_{i_j} = \sum_{t=\ell_j}^{\ell_{j+1}-1} \delta_t$, *and* $y_{i_j} = 1$ *for* $j = 1, \ldots, q$.
- $x_t = 0$ *and* $y_t \in \{0, 1\}$ *for* $t \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_q\}$
- $r_t = \sum_{\ell_j}^{t} \delta_k$, *and* $s_t = 0$ *for* $t \in \{\ell_j, \ldots, i_j - 1\}$ *for all* $j = 1, \ldots, q$.
- $s_t = \sum_{k=t+1}^{\ell_{j+1}-1} \delta_k$, *and* $r_t = 0$ *for* $t \in \{i_j, \ldots, \ell_{j+1} - 1\}$ *for all* $j = 1, \ldots, q$.

Moreover, $\operatorname{conv}(X^{BL})$ *has* $n - 1$ *extreme rays, one for each* $t = 1, \ldots, n - 1$*:*

$$s_t = r_t = 1,\ s_j = r_j = 0 \text{ for } j \neq t,\ x_j = y_j = 0 \text{ for all } j = 1, \ldots, n.$$

The following extended formulation is based on solving ULSB by computing a shortest path in an appropriately defined network (see Figure 1):

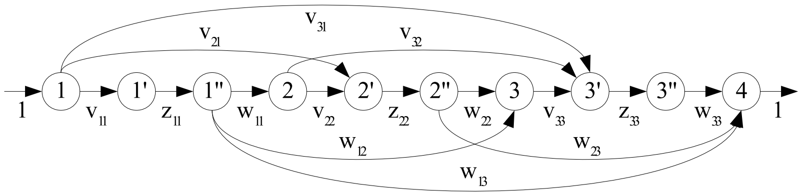

Figure 1: The network of Lemma 1 for $n = 3$.

Lemma 1 *([31]) The optimum value of ULSB equals the optimum value of the following mathematical program.*

$$L^{SP} = \min \sum_{k=1}^{n} \left(\sum_{\ell=1}^{k-1} a_{k\ell} v_{k\ell} + p_k \delta_k z_{kk} + \sum_{\ell=k+1}^{n} b_{k\ell} w_{k\ell} \right) + \sum_{t=1}^{n} f_t y_t + \sum_{t=1}^{n-1} (h_t + g_t) \lambda_t \tag{7}$$

subject to

$$\begin{array}{ll} \sum_{k=1}^{n} v_{k1} = 1, & \textit{node 1}, \\ -v_{11} + z_{11} = 0, & \textit{node 1'}, \\ -z_{11} + \sum_{\ell=1}^{n} w_{1\ell} = 0, & \textit{node 1''}, \\ \sum_{k=t}^{n} v_{kt} - \sum_{k=1}^{t-1} w_{k,t-1} = 0, & \textit{node } t \quad \textit{for } t \geq 2, \\ -\sum_{\ell=1}^{t} v_{t\ell} + z_{tt} = 0, & \textit{node } t' \quad \textit{for } t \geq 2, \\ -z_{tt} + \sum_{\ell=t}^{n} w_{t\ell} = 0, & \textit{node } t'' \quad \textit{for } t \geq 2, \\ z_{tt} - y_t \leq 0, & \textit{for all } t, \\ y_t \leq 1, & \textit{for all } t, \\ v_{kt}, z_{tt}, w_{kt}, y_t, \lambda_t \geq 0, & \textit{for all } t, \end{array} \tag{8}$$

with $a_{k\ell} = p_k \delta_{\ell,k-1} + \sum_{t=\ell}^{k-1} g_t \delta_{\ell,t}$ *and* $b_{k\ell} = p_k \delta_{k+1,\ell} + \sum_{t=k}^{\ell-1} h_t \delta_{t+1,\ell}$ *and* $\delta_{k,\ell} = \sum_{t=k}^{\ell} \delta_t$. *Moreover, v, w, y, and z take integral values in any basic solution.* □

Clearly, if $h_t + g_t \geq 0$ for all t (i.e., the optimum is finite), then the variables λ_t can be dropped from the above formulation.

4 Problem formulation

We consider a supply chain that provides a single item to its customers. It consists of two decision makers, a *leader* and a *follower*. The leader faces a time varying deterministic external demand d_t^1, $t = 1, \ldots, n$, over a discrete time horizon of n time periods. Departing from the external demands, the leader requests a supply of δ_t, $t = 1, \ldots, n$, units from the follower. The follower in turn solves a lot-sizing problem with demands δ_t set by the leader. It generates a *production plan* that specifies for each period t of the planning horizon the amount to be produced. In those time periods t, when a positive amount $x_t^2 > 0$ is produced, a fixed cost of f_t^2 and a variable cost of $p_t^2 x_t^2$ are incurred. The amount x_t^2 is used to serve the request δ_t along with backlogged requests from previous periods; and the remaining quantity, if any, is kept on stock to satisfy future requests. The associated marginal costs are g_t^2 for backlogging, and h_t^2 for holding stocks. In period t the follower delivers a quantity of x_t^1 to the leader. If the follower backlogs some requests in period t, i.e., $r_t^2 > 0$, then we want $x_t^1 = 0$. On the other hand, if $r_t^2 = 0$, then we want that the total request $\sum_{\tau=1}^{t} \delta_\tau$ up time time point t be delivered to the supplier. Since there are no finite production capacities, this is a reasonable requirement. Therefore, x_t^1 and r_t^2 must satisfy

$$\left.\begin{array}{l} r_t^2 = \sum_{\tau=1}^{t}(\delta_\tau - x_\tau^1) \\ x_t^1 r_t^2 = 0 \end{array}\right\} \text{ for all } t = 1, \ldots, n-1. \tag{9}$$

A delivery of $x_t^1 > 0$ incurs a fixed cost of f_t^1, and a variable cost of $p_t^1 x_t^1$ at the leader. The goods are used to satisfy the demand d_t^1 along with backlogged demands of the leader, and the remaining quantity, if any, is kept on stock. The inventory holding costs and the backlogging costs of the leader are h_t^1 and g_t^1, respectively. All the demands must be satisfied by the end of the horizon, i.e., $\sum_{t=1}^{n} d_t^1 = \sum_{t=1}^{n} x_t^1 = \sum_{t=1}^{n} \delta_t = \sum_{t=1}^{n} x_t^2$. The main problem parameters are summarized below:

$n =$ number of periods of the planning horizon

$d_t^1 =$ external demand of the leader

$p_t^1 =$ marginal production costs of the leader

$h_t^1 =$ marginal holding costs of the leader

$g_t^1 =$ marginal backlogging costs of the leader

$f_t^1 =$ fixed production cost of the leader

$p_t^2 =$ marginal production costs of the follower

$h_t^2 =$ marginal holding costs of the follower

$g_t^2 =$ marginal backlogging costs of the follower

$f_t^2 =$ fixed production cost of the follower

The decision variables of the leader and those of the follower are the following:

$x_t^1 \geq 0$: production quantity of the leader in period t

$s_t^1 \geq 0$: stock of the leader after period t

$r_t^1 \geq 0$: backlog of the leader in period t

$y_t^1 \in \{0,1\}$: indicates leader receives supply in period t

$\delta_t \geq 0$: supply requested by the leader in period t

$x_t^2 \geq 0$: production quantity of the follower in period t

$s_t^2 \geq 0$: stock of the follower after period t

$r_t^2 \geq 0$: backlog of the follower in period t

$y_t^2 \in \{0,1\}$: indicates follower produces in period t

$\beta_t^2 \in \{0,1\}$: indicates follower backlogs in period t

Without loss of generality we assume that $f_t^1, f_t^2 \geq 0$ for all t, and to ensure that the optima are finite, $h_t^1 + g_t^1 \geq 0$ and $h_t^2 + g_t^2 \geq 0$ for all $t = 1, \ldots, n-1$ (cf. [31]).

The optimal solution of the follower depends on the quantities δ_t requested by the leader, while that of the leader heavily depends on the supply received from the follower. Therefore, the leader has to carefully choose the requests δ_t.

Clearly, this is a bilevel optimization problem. In the following we focus on the optimistic case only. Recall that in the optimistic case the leader can always choose an optimal solution of the follower (with respect to the parameters δ_t) which is the most advantageous for him. The pessimistic case is usually harder to solve, as we would have to minimize the maximum of the objective function of the leader.

We set up a mathematical program (*Bilevel-LS*) for modeling the decision problem of the leader in the optimistic case.

$$\text{Minimize} \sum_{t=1}^{n} \left(p_t^1 x_t^1 + f_t^1 y_t^1 + h_t^1 s_t^1 + g_t^1 r_t^1 - g_t^2 r_t^2\right) \tag{10}$$

subject to

$$x_t^1 + s_{t-1}^1 - r_{t-1}^1 = d_t^1 + s_t^1 - r_t^1, \quad t = 1, \ldots, n \tag{11}$$

$$r_t^2 = \sum_{\tau=1}^{t} (\delta_\tau - x_\tau^1), \quad t = 1, \ldots, n \tag{12}$$

$$x_t^1 \leq M y_t^1, \quad t = 1, \ldots, n \tag{13}$$

$$x_t^1 \leq M(1 - \beta_t^2), \quad t = 1, \ldots, n-1 \tag{14}$$

$$s_0^1 = s_n^1 = r_0^1 = r_n^1 = 0, \tag{15}$$

$$x_t^1, r_t^1, s_t^1, \delta_t \geq 0, \quad t = 1, \ldots, n \tag{16}$$

$$y_t^1 \in \{0,1\}, \quad t = 1, \ldots, n \tag{17}$$

$$\begin{pmatrix} y^2 \\ x^2 \\ s^2 \\ r^2 \\ \beta^2 \end{pmatrix} \in \arg\min \left\{ \sum_{t=1}^{n} \left(p_t^2 x_t^2 + f_t^2 y_t^2 + h_t^2 s_t^2 + g_t^2 r_t^2 \right) \mid (19) - (24) \right\} \quad (18)$$

where

$$
\begin{aligned}
& x_t^2 + (s_{t-1}^2 - r_{t-1}^2) = \delta_t + (s_t^2 - r_t^2), && t = 1, \ldots, n && (19)\\
& x_t^2 \leq M y_t^2, && t = 1, \ldots, n && (20)\\
& s_0^2 = s_n^2 = r_0^2 = r_n^2 = 0, && && (21)\\
& r_t^2 \leq M \beta_t^2, && t = 1, \ldots, n-1 && (22)\\
& x_t^2, s_t^2, r_t^2 \geq 0, && t = 1, \ldots, n && (23)\\
& y_t^2 \in \{0, 1\}, && t = 1, \ldots, n && (24)\\
& \beta_t^2 \in \{0, 1\} && t = 1, \ldots, n-1. && (25)
\end{aligned}
$$

In this formulation M is a big constant with $M = \sum_{t=1}^{n} d_t^1$.

The objective is to minimize the leader's total cost minus the penalty received from the follower for backlogged supply. The constraints (11)-(17) and (19)-(24) represent lot-sizing problems with backlogging with the additional constraints (12), (14) and (22). Namely, (12) connects the supply by the follower in period t to the production of the leader in the same period, see equation (9), whereas by (14) and (22), the delivery of $x_t^1 = 0$ if the follower backlogs in period t. Moreover, the optimality condition (18) expresses that the follower chooses its optimal production plan with respect to the quantities requested by the leader.

To avoid pathological cases, we want to ensure that $r_t^2 s_t^2 = 0$ in any optimal solution of the follower. One way to achieve this is to add new constraints $s_t^2 \leq M(1 - \beta_t^2)$, $t = 1, \ldots, n-1$, to the follower's program, but this would increase the number of constraints by $n - 1$. Alternatively, noticing that such a solution can be optimal for the follower only if $g_t^2 + h_t^2 = 0$, we make the following:

Assumption 1 *The backlogging and holding costs of the follower satisfy* $g_t^2 + h_t^2 > 0$ *for* $t = 1, \ldots, n$.

Under this assumption, a solution with $s_t^2 r_t^2 > 0$ cannot be optimal for the follower, since it could decrease its costs by decreasing both of s_t^2 and r_t^2 by the same amount.

5 MIP models

In this section we describe two mathematical programs for solving bilevel lot-sizing problems. In both MIPs we introduce new variables and constraints to encode the structure of optimal solutions of the follower, and to characterize those supply requests δ_t for which the follower has an optimal solution with the selected structure. First, we argue that we may assume that the follower's optimal solution is an extreme solution of ULSB.

Lemma 2 *If the bilevel lot-sizing problem admits an optimal solution, then it admits one in which the follower's solution is an extreme point solution of ULSB.*

Proof We will prove that if the bilevel optimization problem admits an optimal solution, then it has one with $r_t^2 s_t^2 = 0$, $x_t^2 s_{t-1}^2 = 0$, $x_t^2 r_t^2 = 0$, and $s_{t-1}^2 r_t^2 = 0$ for $t = 1, \ldots, n$.

- Since $g_t^2 + h_t^2 > 0$, $r_t^2 s_t^2 = 0$ in any optimal solution of the follower, for $t = 1, \ldots, n$.
- Now we prove $x_t^2 r_t^2 = 0$ and $s_{t-1}^2 r_t^2 = 0$ for all t. Namely, rearranging (19) gives $x_t^2 + s_{t-1}^2 = \delta_t + r_{t-1}^2 - r_t^2 + s_t^2$. Suppose $r_t^2 > 0$, then by the previous point, $s_t^2 = 0$, and $x_t^1 = 0$ by (14). Since $x_t^1 = \delta_t + r_{t-1}^2 - r_t^2$ by (12), we have $x_t^2 + s_{t-1}^2 = \delta_t + r_{t-1}^2 - r_t^2 = x_t^1 = 0$. Since $x_t^2, s_{t-1}^2 \geq 0$, the claim follows.
- Finally, we prove that the optimal solution of the follower can be transformed such that $s_{t-1}^2 x_t^2 = 0$ for all t, while maintaining $r_t^2 s_t^2 = 0$, $s_{t-1}^2 r_t^2 = 0$, and $x_t^2 r_t^2 = 0$ for all t. Namely, let t^* be the smallest index with $s_{t-1}^2 x_t^2 > 0$ in the optimal solution of the follower picked by the leader. Then there exists $\ell < t^*$ with $s_{\ell-1}^2 = 0$, and $x_\ell^2 \geq s_\ell^2 \geq \cdots \geq s_{t^*-1}^2 > 0$. Since the solution is optimal, we have $p_\ell^2 + \sum_{\tau=\ell}^{t^*-1} h_\tau^2 = p_{t^*}^2$. Let $\lambda = s_{t^*-1}^2$. We transform the optimal solution of the follower by decreasing x_ℓ^2, and s_ℓ^2 through $s_{t^*-1}^2$ by λ, and increasing x_{t*}^2 by λ. The new solution is still optimal for the follower, and this transformation has no impact at all on the feasibility or optimality of the solution of the leader. □

To restrict the follower's optimal solutions to extreme point solutions of ULSB, we introduce new constraints to the follower's problem:

$$s_{t-1}^2 \quad \leq \quad M(1 - y_t^2 - \beta_t^2), \quad t = 1, \ldots, n, \tag{26}$$

where we let $\beta_n^2 = 0$ to simplify notation. Notice that $s_{t-1}^2 \geq 0$ and (26) imply $y_t^2 + \beta_t^2 \leq 1$.

Proposition 2 *Any optimal solution of the follower satisfying (26) is an extreme point solution of ULSB.*

Proof We clearly have $r_t^2 s_t^2 = 0$ for all t in any optimal solution of the follower by Assumption 1. Now, if $r_t^2 > 0$, then $\beta_t^2 = 1$ by (22), and thus

$s^2_{t-1} = y^2_t = 0$ by (26), and then $x^2_t = 0$ by (20). Likewise, if $x^2_t > 0$, then $y^2_t = 1$ by (20), whence $s^2_{t-1} = \beta^2_t = 0$ by (26), and then $r^2_t = 0$ by (22). Finally, if $s^2_{t-1} > 0$, then $y^2_t = \beta^2_t = 0$ by (26), and thus $r^2_t = x^2_t = 0$ by (20) and (22), respectively. □

5.1 Formulation MIP-1

Our first formulation is based on expressing the optimality conditions of the follower by connecting a primal formulation, and the dual of an extended formulation of ULSB by a single constraint. To begin, we characterize the set of those supply requests δ of the leader for which the follower has optimal solutions with fixed $\bar{y}^2$ and $\bar{\beta}^2$ values. Firstly, we need some definitions. Let

$$E = \{(y^2, \beta^2) \in \{0,1\}^n \times \{0,1\}^{n-1} \mid y^2_t + \beta^2_t \leq 1\} \tag{27}$$

In terms of ULSB, $(\bar{y}^2, \bar{\beta}^2) \in E$ if and only if each period t is either production ($\bar{y}^2_t = 1$), backlogging ($\bar{\beta}^2_t = 1$), or stocking ($\bar{y}^2_t = \bar{\beta}^2_t = 0$).

For any $(\bar{y}^2, \bar{\beta}^2) \in E$, let $D(\bar{y}^2, \bar{\beta}^2)$ be the set of those supply requests $\delta \in \mathbb{R}^n_+$ such that the follower has an optimal solution $(x^2, y^2, s^2, r^2, \beta^2)$ with $y^2 = \bar{y}^2$, and $\beta^2 = \bar{\beta}^2$. Let $Z^{ULSB}(\delta)$ denote the optimum value of the follower for a given $\delta \in \mathbb{R}^n_+$. Note that $(0_n, 0_{n-1}) \in D(\bar{y}^2, \bar{\beta}^2)$ for any $(\bar{y}^2, \bar{\beta}^2) \in E$.

Lemma 3 *For any $(\bar{y}^2, \bar{\beta}^2) \in E$, $D(\bar{y}^2, \bar{\beta}^2)$ is a convex polyhedron.*

Proof Consider the optimization problem of the follower for fixed $(\bar{y}^2, \bar{\beta}^2) \in E$, and parametrized by δ:

$$Z^{ULSB}_{\bar{y}^2,\bar{\beta}^2}(\delta) = \min \left\{ \sum_{t=1}^{n} \left(p^2_t x^2_t + f^2_t y^2_t + h^2_t s^2_t + g^2_t r^2_t\right) \;\middle|\; \begin{array}{l} (19)\text{-}(23) \\ \beta^2 = \bar{\beta}^2,\ y^2 = \bar{y}^2 \end{array} \right\}.$$

Clearly, $Z^{ULSB}_{\bar{y}^2,\bar{\beta}^2}(\delta) \geq Z^{ULSB}(\delta)$, and $Z^{ULSB}_{\bar{y}^2,\bar{\beta}^2}(\delta) = \infty$ if the supply requests cannot be met with the fixed $\bar{y}^2, \bar{\beta}^2$ parameters. Moreover, $\delta \in D(\bar{y}^2, \bar{\beta}^2)$ if and only if $Z^{ULSB}_{\bar{y}^2,\bar{\beta}^2}(\delta) = Z^{ULSB}(\delta)$. To get a linear characterization of $D(\bar{y}^2, \bar{\beta}^2)$, we need a linear program whose feasible solutions provide *lower bounds* on the optimum value of $Z^{ULSB}(\delta)$ for any δ, and whose optimum value is $Z^{ULSB}(\delta)$, and the δ_t occur in the right hand side only. We start out from the *shortest path* formulation (7)-(8) of [31] in which the demand occurs in the objective function. Notice that this linear program always has a finite optimum for any fixed $\delta \geq 0$. Since $f^2_t \geq 0$, in any optimal solution $z_{tt} = y_t$. Hence, z_{tt} can be substituted out. Taking the dual of the resulting linear program, the δ_t occur only in the right hand side of the constraints. The dual variables are ϕ^2_t, $\phi^2_{t'}$, and $\phi^2_{t''}$ for $t = 1, \ldots, n$, and to simplify notation we define $\phi^2_{n+1} = 0$.

$$D^{SP}(\delta) = \max \phi^2_1 \tag{28}$$

subject to

$$\left.\begin{array}{ll}\phi_t^2 - \phi_{k'}^2 \le a_{k,t}, & k = t, \ldots, n \\ \phi_{t'}^2 - \phi_{t''}^2 \le p_t^2 \delta_t + f_t^2, & \\ \phi_{t''}^2 - \phi_{k+1}^2 \le b_{t,k}, & k = t, \ldots, n\end{array}\right\} \text{ for all } t = 1, \ldots, n. \tag{29}$$

Moreover, by the strong duality of linear programming we have $Z^{ULSB}(\delta) = D^{SP}(\delta)$ for any fixed $\delta \ge 0$. We claim that $\delta \in D(\bar{y}^2, \bar{\beta}^2)$ if and only if the linear system consisting of (19)-(23), (29), $y^2 = \bar{y}^2$, $\beta^2 = \bar{\beta}^2$, and the equation

$$\sum_{t=1}^{n} \left(p_t^2 x_t^2 + f_t^2 y_t^2 + h_t^2 s_t^2 + g_t^2 r_t^2\right) = \phi_1^2 \tag{30}$$

is feasible.

First suppose $\delta \in D(\bar{y}^2, \bar{\beta}^2)$. Then there exists an extreme point optimal solution $(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2)$ of ULSB with $\hat{y}^2 = \bar{y}^2$ and $\hat{\beta}^2 = \bar{\beta}^2$. Let $\bar{\phi}^2$ denote an optimal solution of the dual linear program (28)-(29). Since $(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2)$ is an optimal solution of the ULSB problem, we have $Z^{ULSB}_{\bar{y}^2,\bar{\beta}^2}(\delta) = Z^{ULSB}(\delta) = D^{SP}(\delta)$. Hence, δ, along with $(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2)$, and $\bar{\phi}^2$ satisfies the linear system (19)-(23), (29), (30).

Conversely, given some $\delta \ge 0$, suppose the linear system consisting of (19)-(23), (29) and (30) admits a feasible solution $(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2, \hat{\phi}^2)$ with $\hat{y}^2 = \bar{y}^2$ and $\hat{\beta}^2 = \bar{\beta}^2$. Clearly, $(\hat{x}^2, \hat{y}^2, \hat{s}^2, \hat{r}^2)$ is a feasible solution of the follower's problem of value $\hat{\phi}_1^2$, since it satisfies (30). Therefore, we have $Z^{ULSB}(\delta) \le Z^{ULSB}_{\bar{y}^2,\bar{\beta}^2}(\delta) \le D^{SP}(\delta) = Z^{ULSB}(\delta)$, where the second inequality follows from (30). This implies $\delta \in D(\bar{y}^2, \bar{\beta}^2)$.

To finish the proof, we obtain a linear description of $D(\bar{y}^2, \bar{\beta}^2)$ by projecting out all the variables except the δ_t, $t = 1, \ldots, n$, from the linear system (19)-(23), (29) and (30). □

In order to apply the above result, we can use the extended formulation of $D(\bar{y}^2, \bar{\beta}^2)$ consisting of the inequalities (19)-(23), (29) and (30).

We argue that some of the inequalities satisfied by all feasible solutions of the bilevel lot-sizing problem are implied by others.

Proposition 3 *Inequalities (14) always hold if the follower's solution is an extreme point solution.*

Proof In an extreme point solution of the follower's problem, $r_t^2 > 0$ implies $r_t^2 = \delta_t + r_{t-1}^2$ by Proposition 1. Since (12) holds if and only if $x_\tau^1 = \delta_\tau^2 + r_{\tau-1}^2 - r_\tau^2$ for all τ, $r_t^2 > 0$ implies $x_t^1 = 0$. Hence, (14) is superfluous. □

Now we are ready to describe our first MIP for solving the bilevel lot-sizing problem.

$$\text{MIP-1}: \ \min \left\{ \sum_{t=1}^{n} \left(p_t^1 x_t^1 + f_t^1 y_t^1 + h_t^1 s_t^1 + g_t^1 r_t^1 - g_t^2 r_t^2 \right) \ \middle| \ \begin{array}{l} (11)\text{-}(13), \\ (15)\text{-}(17), \\ (19)\text{-}(25),(26), \\ (29),(30) \end{array} \right\}.$$

We can easily project any feasible solution of MIP-1 to a solution of the bilevel-lot-sizing problem by discarding the values of variables ϕ^2.

Lemma 4 *There is a one-to-one correspondence between the feasible solutions of the bilevel lot-sizing problem and that of MIP-1:*

(i) Any feasible solution of MIP-1 can be projected into a feasible solution of the bilevel lot-sizing problem of the same value.

(ii) Conversely, any feasible solution of the bilevel lot-sizing problem can be extended to a feasible solution of MIP-1 of the same value.

Proof

(i) Let $(\bar{x}^1, \bar{y}^1, \bar{s}^1, \bar{r}^1, \bar{\delta}, \bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2, \bar{\beta}^2, \bar{\phi}^2)$ be a feasible solution of MIP-1. Firstly, we have to verify that $(\bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2)$ is an optimal, extreme point solution of the follower with respect to $\bar{\delta}$. Since $(\bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2, \bar{\beta}^2, \bar{\phi}^2)$ satisfies the extended linear description of $D(\bar{y}^2, \bar{\beta}^2)$ given by Lemma 3, $\bar{\delta} \in D(\bar{y}^2, \bar{\beta}^2)$. Since it also satisfies (26), by Proposition 2 it is an extreme point solution of ULSB. Hence, by Proposition 3, inequalities (14) are satisfied. Therefore, $(\bar{x}^1, \bar{y}^1, \bar{s}^1, \bar{r}^1, \bar{\delta}, \bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2, \bar{\beta}^2)$ is a feasible solution of Bilevel-LS of the same objective function value as that of MIP-1.

(ii) Given a feasible solution $(\bar{x}^1, \bar{y}^1, \bar{s}^1, \bar{r}^1, \bar{\delta}, \bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2, \bar{\beta}^2)$ of Bilevel-LS, since it is optimal for the follower, $(\bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2)$ is an optimal solution of the lot-sizing problem for the follower with respect to demands $\bar{\delta}_t$. Therefore, there exists a solution $\bar{\phi}^2$ of (29) which satisfies (30). Hence, $(\bar{x}^1, \bar{y}^1, \bar{s}^1, \bar{r}^1, \bar{\delta}, \bar{x}^2, \bar{y}^2, \bar{s}^2, \bar{r}^2, \bar{\beta}^2, \bar{\phi}^2)$ is a feasible solution of MIP-1. Clearly, the value of the solution of Bilevel-LS, and that of the corresponding solution of MIP-1 are the same.

□

Theorem 1 *The bilevel lot-sizing problem always has a finite optimum, and the optimum value is that of MIP-1.*

Proof We argue that MIP-1 always has a finite optimum. The statement then follows from Lemma 4. Clearly, the objective function value is bounded from below by $-M\sum_{t=1}^{n}((p_t^1)^- + (h_t^1)^- + (g_t^1)^- + (g_t^2)^+)$, where $(v)^- = -\min\{0,v\}$, and $(v)^+ = \max\{0,v\}$ (this is a very rough estimation). So, it suffices to prove that MIP-1 has at least one feasible solution. Fix $\delta \geq 0$ arbitrarily such that $\sum_{t=1}^{n}\delta_t = \sum_{t=1}^{n} d_t^1$. By Proposition 1, the follower has at least one optimal solution in which the production, stocking and backlogging periods can be represented by a vector from E. Thus we can fix the values of variables y_t^2, β_t^2, as well as the values of variables x_t^2, s_t^2, r_t^2, and ϕ^2. However, the leader's variables can also be fixed with respect to δ_t and β_t^2 in an obvious manner, thus a feasible solution is readily available. □

5.2 Formulation MIP-2

In the second formulation, the follower's optimality conditions are again modeled using a shortest path formulation in a directed graph. The graph consists of n nodes corresponding to the n time periods. For each pair of nodes i,k with $i \leq k$, there are $k-i+1$ parallel edges (i,j,k), $i \leq j \leq k$, all directed from node i to node k. The *length* of an edge (i,j,k) is $c_{i,j,k} = a_{j,i} + f_j + p_j\delta_{i,k} + b_{j,k}$ (for definitions, see Section 3). Clearly, for fixed δ_t values, a shortest path from node 1 to node n provides an optimal solution for the follower's lot-sizing problem.

In our second MIP formulation, we introduce one binary variable for each edge of the directed graph. Let α_{ijk} indicate that the requests δ_t in the interval $i,\ldots,k$ are satisfied by production in period $j \in \{i,\ldots,k\}$. If $\alpha_{ijk} = 1$, then $s_{i-1}^2 = s_k^2 = 0$, and likewise $r_{i-1}^2 = r_k^2 = 0$. In addition, we introduce a variable ϕ_t for each node $t \in \{1,\ldots,n\}$ to represent the length of the shortest path from node 0 node node t, and we also set $\phi_0 = 0$.

Now we are ready to describe the second MIP formulation for solving the bilevel lot-sizing problem.

$$\text{MIP-2}: \quad \min \sum_{t=1}^{n} \left(p_t^1 x_t^1 + f_t^1 y_t^1 + h_t^1 s_t^1 + g_t^1 r_t^1 - g_t^2 r_t^2\right)$$

subject to the constraints (11)-(17), (22) and

$$\beta_t^2 = \sum_{i \le t < j \le k} \alpha_{i,j,k}, \qquad t = 1, \ldots, n-1 \tag{31}$$

$$\sum_{i \le t \le k} \sum_{i \le j \le k} \alpha_{i,j,k} = 1, \qquad t = 1, \ldots, n \tag{32}$$

$$a_{j,i} + f_j + p_j \delta_{i,k} + b_{j,k} + \phi_{i-1} \le \phi_k, \qquad 1 \le i \le j \le k \le n \tag{33}$$

$$a_{j,i} + f_j + p_j \delta_{i,k} + b_{j,k} + \phi_{i-1} \ge \phi_k - M'(1 - \alpha_{i,j,k}), \qquad 1 \le i \le j \le k \le n \tag{34}$$

$$\phi_0 = 0, \tag{35}$$

$$\alpha_{i,j,k} \in \{0,1\}, \qquad 1 \le i \le j \le k \le n.$$

Only the constraints (31)-(35) are new. Clearly, t is a backlogging period if and only if there exists $\alpha_{i,j,k} = 1$ with $i \le t < j$. Hence, $\beta_t^2 = 1$ if and only if t is a backlogging period by (31). Constraints (32) ensure that for each period t precisely one edge of the directed graph is selected to start, finish, or cross it, whence, those edges in a feasible solution with $\alpha_{i,j,k} = 1$ constitute a directed path from node 1 to node n. The edge lengths and node potentials are related in (32). However, if $\alpha_{i,j,k} = 1$, then equality must hold by (33). Notice that M' is a very big constant to accommodate the maximum follower cost over all possible requests of the leader. These two equations together ensure that for any $1-n$ path π determined by the variables $\alpha_{i,j,k}$, the δ_t must be chosen such that π is a shortest $1-n$ path with respect to the arc lengths $c_{i,j,k}$.

MIP-2 is intuitively a simpler, and a more natural formulation of the bilevel lot-sizing problem than MIP-1. But as we will see in the computational evaluation, it is harder to solve to optimality than MIP-1. A reason why MIP-2 is weak are the constraints (34), which involve the very big constant M'.

6 New inequalities for cutting off suboptimal solutions of ULSB

In this section we derive a new inequality for ULSB which does not cut off any optimal solution, but may cut off suboptimal ones.

Let Z_t denote the minimum cost incurred by backlogging a unit of production from period t to a later period, that is $Z_t = \min_{u \ge t+1}(p_u + \sum_{v=t}^{u-1} g_v)$. Notice that Z_t does not carry the fixed cost of production.

Lemma 5 *The following inequalities are satisfied by all optimal solutions of ULSB:*

$$(Z_t - p_t)r_t \leq f_t, \text{ for } t = 1, \ldots, n-1. \quad (36)$$

Proof Proof. Let (x, y, s, r) be any optimal solution. We may assume that it satisfies the condition of Lemma 1. If $r_t = 0$, then the inequality trivially holds. So assume that a positive amount of r_t is backlogged in period t. Then, the total cost associated with this amount is at least $Z_t r_t$, which corresponds to the variable cost of backlogging it until some period u, and producing it in period u. On the other hand, if this amount were produced in period t instead, then the associated cost would be at most $f_t + p_t r_t$. Hence, if $Z_t r_t > f_t + p_t r_t$, the solution is not optimal. Namely, let $k > t$ be the first period after t in which a positive amount $x_t > 0$ is produced (since $r_t > 0$, such a time period exists). Then we define another solution (x', y', s', r') of smaller cost: $x'_j = x_j$ for $j \neq k$ and $j \neq t$, $x'_t = r_t$, and $x'_k = x_k - r_t$; $r'_j = r_j - r_t$ for $j = t, \ldots, k-1$, and $r'_j = r_j$ otherwise; $y'_t = 1$, $y'_k = 1$ if and only if $x'_k > 0$, and $y'_j = y_j$ otherwise, $s' = s$. Clearly, this is a feasible solution. Let C' denote the cost of solution (x', y', s', r') and C that of solution (x, y, s, r). We can express the change in the objective function value as follows:

$$C' - C = f_t + p_t r_t - (p_k + \sum_{v=t}^{k-1} g_v)r_t - f_k(1 - y'_k) \leq f_t + p_t r_t - Z_t r_t < 0.$$

Here, the first equation follows from the fact that $y_t = 0$, since $r_t > 0$, and the definitions; the second inequality from the definition of Z_t and from the assumption $f_k \geq 0$; the third inequality from the assumption $Z_t r_t > f_t + p_t r_t$. Therefore, in an optimal solution, we must have $Z_t r_t \leq f_t + p_t r_t$, which is equivalent to the statement of the Lemma. □

A similar statement can be made about stocking costs. Let S_t denote the minimum cost incurred by stocking a unit of production from some period $u < t$ until period t, that is $S_t = \min_{1 \leq u < t}(p_u + \sum_{v=u}^{t-1} h_v)$. Notice that S_t does not carry the fixed cost of production.

Lemma 6 *The following inequality is satisfied by any optimal solution of ULSB:*

$$(S_t - p_t)s_{t-1} \leq f_t, \text{ for } t = 2, \ldots, n. \quad (37)$$

The proof is analogous to that of Lemma 5. We can apply these inequalities to the follower's problem, and we will call (36) and (37) the *R2* and *S2* inequalities, respectively.

7 Computational experiments

Computational experiments have been performed to evaluate and compare the efficiency of the proposed MIP models. Five variants of the models were considered: MIP-1 without the additional cuts; and a variant with both of the *R2* and the *S2* cuts, called MIP-1CC; MIP-2 without the additional cuts; and finally, a variant of MIP-2 with the *R2* cut, called MIP-2C. The other possible combinations were ignored because they were found less efficient in preliminary experiments. The models have been implemented in [18], using the Mosel programming language. A set of random problem instances has been generated with four different problem sizes, $n \in \{10, 15, 20, 25\}$. Generating 100 instances with each value of n resulted in 400 problem instances altogether. The parameters were randomized as follows:

$$\begin{array}{llll} f_t^1 \leftarrow U[100, 200] & p_t^1 \leftarrow U[1, 5] & h_t^1 \leftarrow U[2, 20] & g_t^1 \leftarrow U[4, 40] \\ f_t^2 \leftarrow U[250, 1000] & p_t^2 \leftarrow U[2, 10] & h_t^2 \leftarrow U[1, 10] & g_t^2 \leftarrow U[2, 20] \\ d_t \leftarrow U[0, 100] & & & \end{array}$$

where $U[a, b]$ is the uniform random distribution over the integers in interval $[a, b]$. The experiments were run on an Intel Xeon X5650 2.67GHz computer under a Debian 6.0 operating system. The time limit was set to 1200 seconds per problem instance for $n \in \{10, 15, 20\}$.

Table 2 presents the experimental results. All figures in the table are combined results over the instances with the given problem size. Column *opt* displays the number of instances solved to optimality out of 100 using the given model. Columns *UB gap* (*LB gap*) contain the maximum and average upper (lower) bound gaps in percent, respectively. For each individual instance, the upper bound gap was computed as $100\frac{UB-LB^*}{UB}$, where UB is the upper bound found by the given MIP model, and LB^* is the best lower bound known for the instance, found by any of the approaches. Similarly, the lower bound gap was calculated as $100\frac{UB^*-LB}{UB^*}$, with UB^* and LB defined analogously. Columns *time* present the maximum and the average computation times.

The results show that model MIP-1 clearly dominates MIP-2 from all aspects. MIP-1 can solve to optimality all instances with $n \leq 30$ and most instances with $n = 40$. On the other hand, larger problems are challenging for this model as well. Using cuts is only beneficial on large instances with $n = 50$ time periods. In fact, among the largest instances, 69 is solved to optimality when cuts are added to the model, in contrast to the 58 instances solved without cuts. The average as well as the maximum computation time rapidly increases over 25 activities.

Model MIP-2 could solve only the smallest instances, with $n = 10$, while it found only a few optimal solutions for medium-sized problems (18 or 57 optimal solutions by MIP-2 and MIP-2C, respectively, for $n = 15$), and timed out on all of the larger instances. Even for the instances that could be solved, the computation times were an order of magnitude larger than with MIP-1. For the instances not solved to optimality, the model without the additional cuts, MIP-2, finished with an extremely large lower bound

Table 2: Experimental results.

		opt	LB gap (%)		UB gap (%)		time (sec)	
			max	avg	max	avg	max	avg
MIP-1	$n = 10$	100	0.00	0.00	0.00	0.00	0.79	0.27
	$n = 15$	100	0.00	0.00	0.00	0.00	1.63	0.59
	$n = 20$	100	0.00	0.00	0.00	0.00	12.46	1.38
	$n = 25$	100	0.00	0.00	0.00	0.00	37.67	4.38
	$n = 30$	100	0.00	0.00	0.00	0.00	224.76	14.57
	$n = 40$	95	17.68	0.57	17.68	0.49	1200.00	215.61
	$n = 50$	58	15.36	2.45	15.17	1.82	1200.00	675.78
MIP-1CC	$n = 10$	100	0.00	0.00	0.00	0.00	0.82	0.28
	$n = 15$	100	0.00	0.00	0.00	0.00	1.68	0.60
	$n = 20$	100	0.00	0.00	0.00	0.00	13.00	1.41
	$n = 25$	100	0.00	0.00	0.00	0.00	28.72	3.96
	$n = 30$	100	0.00	0.00	0.00	0.00	147.66	13.40
	$n = 40$	94	18.82	0.52	17.68	0.48	1200.00	226.91
	$n = 50$	69	14.45	1.96	14.45	1.80	1200.00	617.99
MIP-2	$n = 10$	100	0.00	0.00	0.00	0.00	35.65	17.93
	$n = 15$	18	29.52	5.42	932.61	175.21	1200.00	1152.12
	$n = 20$	0	70.73	43.06	2974.59	1515.28	1200.00	1200.00
MIP-2C	$n = 10$	100	0.00	0.00	0.00	0.00	46.83	10.57
	$n = 15$	57	12.25	1.03	85.12	14.58	1200.00	927.95
	$n = 20$	0	59.32	17.79	192.80	108.28	1200.00	1200.00

gap: the maximum gap was nearly 3000%, while the average gap was above 1500% for $n = 20$. Adding the cut to the model could significantly improve both the upper and the lower bounds, resulting in gaps often 3 (UB) or 10 (LB) times smaller for MIP-2C than for MIP-2 with $n = 20$, but still considerably weaker than the results achieved by the variants of MIP-1.

8 Conclusions

In this paper we have developed exact solutions methods for the bilevel uncapacitated lot-sizing problem with backlogs. The novelty of our approach lies in the modeling of the optimality conditions of the follower by using two formulations for the same problem, and thus we can avoid the use of extra binary variables to model complementarity conditions which is a standard technique in bilevel optimization.

The capacitated version of the problem is even more difficult, even if the follower's problem is a constant capacity lot-sizing problem with backlogs (CC-LSB). Although CC-LSB is polynomially solvable, to apply the technique of this paper, one needs an (extended) formulation for CC-LSB in which the demands occur in the objective function. The extended formulation of [39] is not appropriate for our purposes, since the demands occur in the right hand side.

Finally, our approach may be suitable for solving other bilevel optimization problems where the follower's problem admits an extended formulation in which the parameters imposed by the leader occur only in the objective function, and the optimal solutions have nice structural properties.

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